

# DECAY ESTIMATES FOR ONE-DIMENSIONAL WAVE EQUATIONS WITH INVERSE POWER POTENTIALS

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**ABSTRACT.** We study the one-dimensional wave equation with an inverse power potential that equals  $\text{const.}x^{-m}$  for large  $|x|$  where  $m$  is any positive integer greater than or equal to 3. We show that the solution decays pointwise like  $t^{-m}$  for large  $t$ , which is consistent with existing mathematical and physical literature under slightly different assumptions (see e.g. [1], [4], [6]).

Our results can be generalized to potentials consisting of a finite sum of inverse powers, the largest of which being  $\text{const.}x^{-\alpha}$  where  $\alpha > 2$  is a real number, as well as potentials of the form  $\text{const.}x^{-m} + O(x^{-m-\delta_1})$  with  $\delta_1 > 3$ .

## 1. INTRODUCTION

There is an extensive literature - both mathematical and physical- on the study of decay for wave equations and Schrödinger equations with a potential (see e.g. [6] for a survey). In the physical community the corresponding problem goes by the name of tails, and physicists predicted the decay of solutions to wave equations on the line with potentials decaying like  $|x|^{-\alpha}$  as  $|x| \rightarrow \infty$  based on nonrigorous and numerical methods, see for example [1, 2].

Mathematically, a recent study by R. Donniger and W. Schlag ([4]) showed that for potentials decaying like  $|x|^{-\alpha}$  where  $2 < \alpha \leq 4$  with no bound state or zero energy resonance, the solution  $\psi$  to the one-dimensional wave equation

$$(1) \quad \frac{\partial^2 \psi(x, t)}{\partial t^2} - \frac{\partial^2 \psi(x, t)}{\partial x^2} + V(x)\psi(x, t) = 0$$

is bounded by  $t^{-\alpha}$  for large  $t$ .

The purpose of this paper is to give sharp estimates for the decay of  $\psi$  where  $V(x) = \text{const.}x^{-m}$  for large  $|x|$  (the constants are allowed to be different for positive and negative  $x$ ) where  $m \in \mathbb{N}$  and  $m \geq 3$ . The result is consistent with [4] and confirms the predictions by physicists.

## 2. SETTING AND MAIN RESULTS

We analyze the wave equation (1) under the assumptions:

**Assumption 1.** (i) The potential  $V$  is such that the one-dimensional Schrödinger operator  $A := -\frac{d^2}{dx^2} + V(x)$  has no bound states and no zero energy resonances.

(ii)  $V$  is  $m+2$  times differentiable.

(iii) As  $x \rightarrow \pm\infty$  we have  $V(x) = \text{const.}\pm x^{-m}$  where  $m \in \mathbb{N}$  and  $m \geq 3$ .

With  $\psi_0(x) = \psi(x, 0)$ , and  $\psi_1(x) = \frac{\partial \psi(x, 0)}{\partial t}$  the solution to (1) (cf. [4]) can be written as

$$\psi(t) = \cos(t\sqrt{A})\psi_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}\psi_1$$

Our main results are

**Theorem 1.** *Under Assumption 1 we have*

$$\begin{aligned} \frac{\sin(t\sqrt{A})}{\sqrt{A}}\psi_1 &= \hat{r}_1(x)\langle t \rangle^{-m} + \langle t \rangle^{-m}R_1(x, t) \\ \cos(t\sqrt{A})\psi_0 &= \hat{r}_0(x)\langle t \rangle^{-m-1} + \langle t \rangle^{-m-1}R_0(x, t) \end{aligned}$$

where

$$\begin{aligned} \|\langle x \rangle^{-2}\hat{r}_j(x)\|_\infty &\lesssim \|\langle x \rangle^2\psi_j(x)\|_1, \quad j = 0, 1 \\ \|\langle x \rangle^{-m-2}R_1(x, t)\|_\infty &\leq \|\langle x \rangle^{m+2}\psi_1(x)\|_1 \\ \|\langle x \rangle^{-m-3}R_0(x, t)\|_\infty &\leq \|\langle x \rangle^{m+3}\psi_0(x)\|_1 + \|\langle x \rangle^{m+3}\psi'_0(x)\|_1 \end{aligned}$$

Here  $\langle x \rangle := (1+x^2)^{1/2}$ , the infinity norm for  $R_{0,1}(x, t)$  is in both  $x$  and  $t$ , and  $\lim_{t \rightarrow \infty} R_{0,1}(x, t) = 0$ . Moreover,  $\hat{r}_j(x)$  is nonzero for generic initial data (cf. Remark 6).

In Section 6 we discuss generalizations where  $V$  is a sum of inverse powers, and an extension of results of the type in [6]. The special case  $m = 2$  will also be briefly discussed in Note 1 below.

The basic strategy we use is to take the Laplace transform in  $t$  of (1) and study the solutions of the transformed equation. Laplace transformability is shown in Proposition 14 in the Appendix; its existence does not require Assumption 1 (i); the result of Theorem 1 is however contingent on it.

By taking the  $t$ -Laplace transform of (1) we obtain the ODE

$$(2) \quad \hat{\psi}''(x, \varepsilon) = (V(x) + \varepsilon^2) \hat{\psi}(x, \varepsilon) + \psi_1(x) + \varepsilon \psi_0(x)$$

where

$$\hat{\psi}(x, \varepsilon) = \int_0^\infty e^{-\varepsilon t} \psi(x, t) dt$$

The analysis relies on properties of Jost functions (for  $x > 0$  this is defined by the behavior (5)) solving the homogeneous equation

$$(3) \quad y''(x) = (V(x) + \varepsilon^2) y(x)$$

**Note 1.** For  $V(x) = a/x^m$  and  $m = 1, 2$  the equation can be solved in terms of special functions.

For  $V(x) = a/x^2$ , the solution that decays like  $e^{-\varepsilon x}$  as  $x \rightarrow \infty$  is given in terms of the modified Bessel function  $K$  as

$$y^+ = \sqrt{2\varepsilon x/\pi} K_\alpha(\varepsilon x); \quad \alpha = \sqrt{a + 1/4}$$

For small  $\varepsilon$  and fixed  $x$ ,  $y^+$  has the form

$$(4) \quad C_1(x) \varepsilon^{1+\alpha} A_1(\varepsilon) + C_2(x) \varepsilon^{1-\alpha} A_2(\varepsilon)$$

with  $A_1, A_2$  analytic.

For  $m \geq 3$ , the Jost functions have asymptotic expansions in integer powers of  $\varepsilon$  and  $\varepsilon \ln \varepsilon$ . By symmetry it is sufficient to study the case  $x > 0$ . We thus analyze the Jost function given, for  $\varepsilon > 0$ , by

$$(5) \quad y(x) \sim e^{-\varepsilon x} (1 + s(x; \varepsilon))$$

where  $s(x; \varepsilon)$  is an  $o(1)$  power series in  $1/x$ , as  $x \rightarrow \infty$ . The function  $s$  satisfies

$$(6) \quad s'' - 2\varepsilon s' - V(x)s = V(x)$$

It is easy to see that an  $o(1)$  solution of (6) exists and is unique.

### 3. PROPERTIES OF $s(x; \varepsilon)$

We first assume  $V$  is  $m + 2$  times differentiable and

$$V(x) = \begin{cases} v_1 x^{-m}, & x \geq x_+ > 1 \\ v_2 x^{-m}, & x \leq x_- < -1 \end{cases}$$

By rescaling  $x$  and  $\varepsilon$  one can make  $v_{1,2} = \pm 1$ .

To analyze the behavior of  $s(x; \varepsilon)$  for small  $\varepsilon$  and  $x \geq x_+$ , it is convenient to study its inverse Laplace transform, this time in  $x$ , to regularize the behavior at turning points. Inverse Laplace transformability does not need to be proved at this stage, since at the end we show that the Laplace transform of the solution to the dual equation solves (6). We let  $\hat{s} =: H(q)/[q(q + 2\varepsilon)]$  be the formal inverse Laplace transform of  $s$  ( $x \mapsto q$  with  $\text{Re } x > x_+$ ), and obtain

$$(7) \quad H(q) = \frac{v_1}{(m-1)!} q^{m-1} + \mathcal{P}^m \frac{v_1 H(q)}{q(q + 2\varepsilon)}$$

where  $\mathcal{P}F(q) = \int_0^q F(u) du$ . With the change of variable  $q = \varepsilon \tau$ ,  $H(q) = F(\tau)$ , we obtain

$$(8) \quad F(\tau) = \frac{v_1 \varepsilon^{m-1} \tau^{m-1}}{(m-1)!} + \varepsilon^{m-2} \mathcal{P}^m \frac{v_1 F(\tau)}{\tau(\tau + 2)}$$

We show that  $F(\tau)$  has an asymptotic expansion in powers of  $\varepsilon$ ,  $\tau^{-1}$  and  $\tau^{2-m} \ln \tau$  for small  $\varepsilon$  and large  $\tau$ .

**Lemma 1.** (i) The function  $\hat{s}$  has a Laplace transform in  $q$ , and  $F$  is analytic for  $\operatorname{Re} \tau > -2$  and entire in  $\varepsilon$  and has the convergent expansion

$$(9) \quad F(\tau) = \varepsilon^{m-1} F_{m-1}(\tau) + \sum_{j \geq m} \varepsilon^j F_j(\tau)$$

where  $j_m = (m-2)(j-m+1) + m-1$ ,  $F_{m-1}(\tau) = v_1 \tau^{m-1}/(m-1)!$ , while for  $j \geq m$  we have

$$(10) \quad F_j(\tau) = \tau^{j_m} \sum_{n=0}^{j-m+1} \tau^{-n(m-2)} (\ln(\tau))^n W_n(\tau^{-1}); \quad W_n(z) \text{ analytic for } |z| < \frac{1}{2}$$

Furthermore,  $F_j$  is analytic in  $\mathbb{H}_1 := \{z \in \mathbb{C} : \arg z \in (-\pi/4, 5\pi/4), z \neq 0\}$  with

$$(11) \quad |F_j| \lesssim \frac{|\tau|^{j_m}}{((j_m - m + 1)!)^{\frac{m}{m-2}}} \quad \text{for } \tau \in \mathbb{H}_1$$

(ii) In fact we only need to keep the first few terms of (10) and estimate the remainder. For  $|\tau| \geq 3$  we write

$$(12) \quad F_j(\tau) = F_j^{[0]}(\tau) + \tau^{j_m-m} G_j(\tau);$$

$$F_j^{[0]}(\tau) := \tau^{j_m} \left( \sum_{n=0}^{m-1} a_{n,0}^{[j]} \tau^{-n} + \sum_{k=1}^2 \sum_{n=0}^{2-k} a_{k(m-2)+n,k}^{[j]} \tau^{-k(m-2)-n} (\ln \tau)^k \right);$$

We have  $a_{l,k}^{[j]} = 0$  if  $j < m+k-1$  or  $l > m$ , and the following estimates hold: for some  $c_1 > 0$  (independent of all indices above)

$$(13) \quad |a_{n,l}^{[j]}| \leq \frac{c_1^n}{((j_m - 2m + 2)!)^{\frac{m}{m-2}}}, \quad \sup_{\tau \in \mathbb{H}_1} |(|\ln \tau| + 1)^{-3} G_j(\tau)| \leq \frac{c_1 j}{((j_m - 2m + 1)!)^{\frac{m}{m-2}}}$$

*Proof.* We analyze the case  $v_1 = 1$ ; if  $v_1 = -1$ , the arguments are very similar. We look for solutions to (8) which are  $O(\varepsilon^{m-1} \tau^{m-1})$  for small  $\tau$ . Consider the space  $\mathcal{B}$  of functions of the form  $f(\tau) = \tau^{m-1} \tilde{G}(\tau)$  where  $\tilde{G}$  is analytic for, say,  $|\tau| < \tau_0$  for arbitrarily large  $\tau_0 > 0$  with the norm  $\|f\| = \sup_{|\tau| < \tau_0} |\tilde{G}(\tau)|$ . We see that this is a Banach space, and eq. (8) is contractive in  $\mathcal{B}$ . The solution of (8) is unique, and it is analytic for small  $\tau$ . As a differential equation (8) reads

$$(14) \quad F^{(m)} = \frac{\varepsilon^{m-2} F}{\tau(\tau+2)}$$

The argument above, or Frobenius theory, shows that (14) also has a unique solution which is of the form  $\frac{1}{(m-1)!} \varepsilon^{m-1} \tau^{m-1} (1 + o(1))$  for small  $\tau$ . The solution is obviously analytic for  $\operatorname{Re} \tau > -2$ , since the only singularities of equation (8) are  $\tau = 0$  and  $\tau = -2$ , and it is entire in  $\varepsilon$  for  $\operatorname{Re} \tau > -2$ , since the equation depends analytically in  $\varepsilon$ .

By standard ODE asymptotic results [7] we see that any solution of (14) is uniformly bounded in  $\mathbb{C}$  by

$$(15) \quad C(\varepsilon) |\tau|^{\frac{m-1}{m}} e^{\frac{m}{m-2} |\varepsilon|^{1/m} |\tau|^{1-2/m}}$$

for some  $C(\varepsilon) > 0$ . This ensures the necessary (sub)exponential bounds for taking the Laplace transform in  $\varepsilon$ .

We now look for solutions of (14) in the form

$$(16) \quad F = \frac{\varepsilon^{m-1} \tau^{m-1}}{(m-1)!} + \sum_{j \geq m} \varepsilon^j F_j$$

and we show that the expansion (16) is convergent.

The functions  $F_j$  satisfy the recurrence

$$(17) \quad F_{j+1} = \mathcal{P}^m \frac{F_j}{\tau(\tau+2)}, \quad j \geq m-1; \quad F_{m-1}(\tau) = \tau^{m-1}/(m-1)!$$

For now, we take  $\tau$  in  $\mathbb{H}_1$ . It can be checked by induction that  $F_j$  are analytic in  $\mathbb{H}_1$  and at zero, and since  $|\tau^{j_m}| \leq |\tau^{j_m-1}(\tau+2)|$  in  $\mathbb{H}_1$  and

$$|\mathcal{P}^m \tau^{j_m-2}| = \frac{|\tau|^{j_m+m-2}}{\prod_{k=0}^{m-1} (j_m-1+k)} \leq \left( \frac{(j_m-m+1)!}{((j+1)_m-m+1)!} \right)^{\frac{m}{m-2}} |\tau|^{(j+1)_m}$$

(11) follows by induction. The last inequality above comes from the fact that

$$(18) \quad \left( \prod_{k=0}^{m-1} (j_m-1+k) \right)^{m-2} \geq (j_m-1)^{m(m-2)} \geq \left( \prod_{k=0}^{m-3} (j_m-m+2+k) \right)^m$$

It follows that the series (16) converges uniformly on any compact set in  $\mathbb{H}_1$ . Moreover, we see that the function series

$$(19) \quad H(q) = \frac{q^{m-1}}{(m-1)!} + \sum_{j \geq m} \varepsilon^{j_m} F_j(q/\varepsilon)$$

also converges uniformly in  $q$  on any compact set in  $\mathbb{H}$ . Existence of the Laplace transform of  $\frac{H(q)}{q(q+2\varepsilon)}$  follows from the bound (15) for  $F$ .

We write (8) as

$$(20) \quad \tau(\tau+2)F_{j+1}^{(m)} = F_j, \quad j \geq m-2; \quad F_{m-2} = 0$$

Note that  $F_{m-1}$  is explicit (see (17)). Let  $Lg = \tau(\tau+2)g^{(m)}(\tau)$ . Eq. (20) implies

$$(21) \quad L^{j-m+2}F_j = 0; \quad j \geq m$$

**Note 2.** The indicial polynomial of (21) at infinity is

$$(22) \quad \prod_{n=0}^{j-(m-1)} \prod_{n'=0}^{m-1} (\lambda - n' - n), \quad j \geq m$$

with the convention that a product is one if the index set is empty, and (22) implies (10). Eq. (22) follows from

$$(23) \quad L\tau^\lambda = \tau^{\lambda-m+2}[\lambda(\lambda-1)\cdots(\lambda-m+1) + O(\tau^{-1})]$$

(ii) The existence of an asymptotic expansion of the form (12) follows from (10). It remains to estimate the coefficients and the remainder (which we do recursively), for which we can assume  $j_m \geq 2m-2$  since for  $j_m < 2m-2$  the result follows directly from (10).

We have

$$(24) \quad L(t^n \ln^l(t)) = t^{n-1} \ln^l t (1 + 2t^{-1}) \left( n(n-1)\cdots(n-m+1) + \sum_{l=1}^m P_l(\ln t)^{-l} \right)$$

where  $P_l$  are polynomials of degree at most  $m-1$  in  $n$  and  $m$  in  $l$ . Substituting (9) in (20) using the notation in (12) and taking  $a_{n,l}^{[j]} = ((j_m-2m+2)!)^{-\frac{m}{m-2}} A_{n,l}^{[j]}$  we get the following recurrence for  $0 \leq l \leq 3$  with  $(m-2)l \leq n \leq m-1$

$$(25) \quad A_{n,l}^{[j]} - C_{n,l}^{[j]} A_{n,l}^{[j-1]} + \sum_{J_{n,l}} C_{n',l';n,l}^{[j]} A_{n',l'}^{[j]} = 0$$

where  $J_{n,l}$  consists of indices earlier than  $n, l$ :  $J_{n,l} = \{(n, l') : l' > l\} \cup \{(n-1, l') : 0 \leq l' \leq 2\}$ . In (25) we have  $0 < C_{n,l}^{[j]} < 1$  and for some  $c_4 > 0$  and all  $n, n', l, l', j$  we have  $|C_{n',l';n,l}^{[j]}| < c_4$ . Solving for  $A_{n,l}^{[j]}$  in the order  $n = 0, 1, \dots, m-1$  and for a fixed  $n$  in the order  $l = 2, 1, 0$ , the first inequality in (13) follows inductively on  $j$ .

**Note 3.** Let  $R_j = \tau^{j_m-m-1}G_j$ . Then,  $R_j$  satisfies the recurrence

$$(26) \quad \tau(\tau+2)R_j^{(m)} = R_{j-1} + \tau^{j_m-2m+1}p_1(\ln \tau)$$

where  $p_1$  is a quadratic polynomial with coefficients bounded by

$$\frac{c_2}{((j-1)_m - 2m + 2)!^{\frac{m}{m-2}}}$$

for some  $j$ -independent  $c_2$ .

Equation (26) simply follows by writing  $F_j(\tau) = F_j^{[0]} + R_j(\tau)$ , calculating the finite sum  $LF_j^{[0]}$  explicitly using (23) and (24) and estimating the coefficients of  $p_1$  using the first inequality in (13).

*Proof of the last inequality in (13).* Since  $j_m = (j-1)_m + m - 2$ , we have by (26)

$$(27) \quad |R_j| = |\mathcal{P}^m \frac{\tau^{(j-1)_m - m - 1} (G_{j-1}(\tau) + p_1(\ln \tau))}{\tau + 2}| \leq \mathcal{P}^m |\tau^{(j-1)_m - m - 2} (\tau G_{j-1}(\tau) + p_1(\ln \tau))|$$

Also, by direct integration we have

$$(28) \quad \int_0^\tau |t^n \ln^l t| dt \leq c_3 (n+1)^{-1} |\tau^{n+1}| (|\ln^l \tau| + 1); \quad l = 0, 1, 2; \quad n > 0$$

for some  $c_3$ . The rest follows from (27) and (28) by induction on  $j$ , noting that (cf. also (18))

$$(29) \quad \frac{1}{\prod_{k=0}^{m-1} ((j-1)_m - m - 1 + k)} \leq \left( \frac{((j-1)_m - 2m + 1)!}{(j_m - 2m + 1)!} \right)^{\frac{m}{m-2}}$$

□

**Lemma 2.** (i) For  $|q| \geq 3|\varepsilon|^{\frac{1}{m-2}}$  we have the expansion

$$(30) \quad H(q) = q^{m-1} H_{0,0}(q) + \sum_{k=1}^{m-1} \varepsilon^k q^{2m-3-k} H_{k,0}(q) + \sum_{n=1}^2 \sum_{k=0}^{2-n} \varepsilon^{n(m-2)+k} q^{m-1-k} H_{k,n}(q) \ln^n(q/\varepsilon) + \varepsilon^m \tilde{R}(q/\varepsilon, q)$$

where  $H_{i,j}(q)$  ( $j \leq 2$ ) are analytic in  $q$  with sub-exponential growth for large  $q$ ,  $H_{0,2} = 0$  if  $m > 3$ , and  $|\partial^{(k+l)} \tilde{R}(u, v) / \partial u^k \partial v^l| \lesssim (|\ln u| + 1)^3 |u|^{-k}$  for  $\operatorname{Re} u \geq 0$ ,  $|v| < \text{const.}$ , and  $0 \leq k + l \leq m + 1$ .

(ii) For  $|q| \leq 3|\varepsilon|^{\frac{1}{m-2}}$  and  $\operatorname{Re}(q/\varepsilon) \geq 0$   $H(q)$  is analytic in  $q$ , entire in  $\varepsilon$ , and  $|H(q)| \lesssim |q|^{m-1}$ .

*Proof.* (i) Recall that  $q = \varepsilon\tau$ ,  $H(q) = F(\tau)$ , and  $a_{l,k}^{[j]} = 0$  if  $j < m + k - 1$  by Lemma 1. We thus substitute  $\tau = q/\varepsilon$  in (12) and obtain (29) by collecting coefficients of powers of  $\varepsilon$  and  $\ln(q/\varepsilon)$ . We define

$$\begin{aligned} q^{m-1} H_{0,0}(q) &= \frac{q^{m-1}}{(m-1)!} + \sum_{j \geq m} q^{j_m} a_{0,0}^{[j]} \\ q^{2m-3-k} H_{k,0}(q) &= \sum_{j \geq m} a_{k,0}^{[j]} q^{j_m - k} \quad (1 \leq k \leq m-1) \\ q^{m-1-k} H_{k,n}(q) &= \sum_{j \geq m+n-1} a_{n(m-2)+k,n}^{[j]} q^{j_m - (n(m-2)+k)} \quad (1 \leq n \leq 2, 0 \leq k \leq 2-n) \\ \tilde{R}(q/\varepsilon, q) &= \sum_{j_m \geq m} q^{j_m - m} G_j(q/\varepsilon) \end{aligned}$$

Convergence of the series is ensured by (13). Here the powers  $q^n$  on the left side represent the lowest power of  $q$  on the right side. Since  $F_j(\tau)$  is analytic in  $\mathbb{H}_1$  (see lemma 1), by (13) and Cauchy's formula we have

$$|G_j^{(k)}(\tau)| \lesssim \frac{j |\tau|^{-k} (|\ln \tau| + 1)^3}{((j_m - 2m + 1)!)^{\frac{m}{m-2}}}$$

for  $\tau \in \mathbb{H}$ . Noting that for  $a > 0$  we have  $(j!)^a > \text{const.}^j \Gamma(aj + 1)$ , (13) implies

$$\sum_{j=m}^{\infty} \frac{|\tau|^{j_m}}{((j_m - 2m + 2)!)^{\frac{m}{m-2}}} \leq \sum_{j=0}^{\infty} \frac{|\tau|^j}{(j!)^{\frac{m}{m-2}}} \lesssim \sum_{j=0}^{\infty} \frac{|c_t \tau|^j}{\Gamma(\frac{m}{m-2}j + 1)} \lesssim e^{\delta|\tau|}, \quad \forall \delta > 0$$

<sup>13</sup> can be replaced by any constant bigger than 2.

Thus sub-exponential growth of  $H_{i,j}$  follows. In fact it is elementary to show that the last sum is bounded by  $e^{\text{const.} \tau^{\frac{m-2}{m}}}$ . The rest of (i) follows from (9) using (11) and (12) to estimate the terms.

(ii) This follows from Lemma 1 (i), especially (9) and (11).  $\square$

**Corollary 3.** (i) For  $|q| \geq 3|\varepsilon|$  we have the expansion

$$(31) \quad \frac{H(q)}{q(q+2\varepsilon)} = \sum_{n=0}^2 \varepsilon^{n(m-2)} \left( \frac{\varepsilon^{m-2} \tilde{H}_{1,n+1}(\varepsilon)}{q+2\varepsilon} + \tilde{H}_{2,n+1}(q, \varepsilon) \right) \ln^n(q/\varepsilon) + \varepsilon^m \frac{\tilde{R}(q/\varepsilon, q)}{q(q+2\varepsilon)}$$

where  $\tilde{H}_{1,j}$  are entire,  $\tilde{H}_{2,j}(q, \varepsilon)$  are entire in  $\varepsilon$ , analytic in  $q$  and have sub-exponential growth for large  $q$ ,  $\tilde{H}_{2,1}(\varepsilon, \varepsilon) = O(\varepsilon^{m-3})$ ,  $\tilde{H}_{1,1}(0) = \frac{(-2)^{m-2}}{\Gamma(m)}$ ,  $\tilde{H}_{k,3} = 0$  if  $m > 3$ , and  $\tilde{R}$  is the same as in Lemma 2.

(ii) For  $|q| \leq 3|\varepsilon|$  and  $\text{Re}(q/\varepsilon) \geq 0$  we have

$$(32) \quad \frac{H(q)}{q(q+2\varepsilon)} = \frac{\varepsilon^{m-2} H_1(\varepsilon)}{q+2\varepsilon} + \varepsilon^{m-3} H_2(q, \varepsilon)$$

where  $H_i$  are analytic in  $q$  and entire in  $\varepsilon$ .

*Proof.* We use (30) by noting that for a function  $f$  analytic in  $q$  and entire in  $\varepsilon$  we have

$$(33) \quad \frac{f(q)}{q+2\varepsilon} = \frac{f(-2\varepsilon)}{q+2\varepsilon} + \frac{f(q) - f(-2\varepsilon)}{q+2\varepsilon} := \frac{\tilde{f}_1(\varepsilon)}{q+2\varepsilon} + \tilde{f}_2(q; \varepsilon)$$

where  $\tilde{f}_2$  is analytic in  $q$  and entire in  $\varepsilon$ . According to (30), we take in (33)

$$qf(q) = q^{m-1} H_{0,0}(q) + \sum_{k=1}^{m-1} \varepsilon^k q^{2m-3-k} H_{k,0}(q)$$

and define  $\varepsilon^{(2-k)(m-2)} \tilde{H}_{k,1} = \tilde{f}_k$  ( $k = 1, 2$ ). For  $k, n = 1, 2$  we take in (33)

$$qf(q) = \sum_{k=0}^{2-n} \varepsilon^k q^{m-1-k} H_{k,n}(q)$$

and define  $\varepsilon^{(2-k+n)(m-2)} \tilde{H}_{k,n} = \tilde{f}_k$  ( $k, n = 1, 2$ ). Sub-exponential growth of  $\tilde{H}_{2,j}$  follows immediately from the sub-exponential growth of  $H_{2,j}(q)$ . In addition  $\tilde{H}_{1,1}(0) = H_{0,0}(0) = -\frac{1}{2} F_{m-1}(-2) = \frac{(-2)^{m-2}}{\Gamma(m)}$  by Lemma 1 and the proof of Lemma 2.

Similarly to obtain (32) we use (33) for  $f(q) = H(q)$  and apply Lemma 2 (ii).  $\square$

**3.1. Asymptotic expansion of  $s(x; \varepsilon)$  for small  $\varepsilon$ .** For small  $\varepsilon$ , the function  $s(x; \varepsilon)$  (cf. (5)) has an asymptotic expansion in powers of  $\varepsilon$  and  $\varepsilon^{m-2} \ln \varepsilon$ . For our purpose we only need a few terms of this expansion.

**Lemma 4.** (i) Let  $\delta > 0$  be arbitrarily small but fixed. We have for  $x \geq x_+$ ,  $\varepsilon \in \mathbb{H}$  and  $|\varepsilon| \leq 1/x$

$$(34) \quad s(x; \varepsilon) = h_1(x) \varepsilon^{m-2} \ln \varepsilon + h_2(x) \varepsilon^{m-1} \ln \varepsilon + h_3(x) \varepsilon^{2m-4} (\ln \varepsilon)^2 + Q(x; \varepsilon)$$

where the smooth functions  $h_j$  satisfy

$$(35) \quad h_1(x) \sim -\frac{(-2)^{m-2}}{\Gamma(m)}, h_2(x) \sim \frac{(-2)^{m-1} x}{\Gamma(m)}, h_3(x) \sim a_0 \text{ as } x \rightarrow \infty$$

where  $a_0$  is a constant and for  $m > 3$  one can take  $h_3(x) = 0$ . Furthermore

$$(36) \quad \sup_{0 \leq k \leq m-1} \left| x^{m-2-k} \frac{\partial^k Q(x; \varepsilon)}{\partial \varepsilon^k} \right| < \infty; \quad \sup_{m \leq k \leq m+1} \left| \varepsilon^{k-m+\delta} x^{-2} \frac{\partial^k Q(x; \varepsilon)}{\partial \varepsilon^k} \right| < \infty$$

The asymptotic formula (34) is twice differentiable in  $x$ , i.e. (36) holds with  $Q$  replaced by  $xQ'$  and  $x^2 Q''$ .

*Proof.* We write

$$(37) \quad s(x; \varepsilon) = \left( \int_0^{3\varepsilon} + \int_{3\varepsilon}^1 + \int_1^\infty \right) \frac{H(q)e^{-qx}}{q(q+2\varepsilon)} dq$$

The first and last terms are estimated easily as follows: the last integral is manifestly analytic in  $\varepsilon$  and decays exponentially in  $x$ , so (36) is obvious. By Corollary 3 the first integral is equal to

$$\varepsilon^{m-2} \int_0^3 \left( \frac{H_1(\varepsilon)}{\tau+2} + H_2(\varepsilon\tau, \varepsilon) \right) e^{-\varepsilon\tau x} d\tau$$

Thus it is analytic in  $\varepsilon$  and satisfies the estimates in (36) by direct differentiation in  $\varepsilon$ , noting that  $|\varepsilon^{m-2}| \lesssim x^{2-m}$ .

To estimate the middle integral in (37) we use (31) to write

$$\int_{3\varepsilon}^1 \frac{H(q)e^{-qx}}{q(q+2\varepsilon)} dq = s_a(x, \varepsilon) + S_2(x, \varepsilon) + S_3(x, \varepsilon)$$

where

$$(38) \quad S_1(x, \varepsilon) = \sum_{k=1}^3 \varepsilon^{k(m-2)} \tilde{H}_{1,k}(\varepsilon) \int_{3\varepsilon}^1 \frac{(\ln(q/\varepsilon))^{k-1} e^{-qx}}{q+2\varepsilon} dq$$

$$(39) \quad S_2(x, \varepsilon) = \sum_{k=0}^2 \varepsilon^{k(m-2)} \int_{3\varepsilon}^1 (\ln(q/\varepsilon))^k \tilde{H}_{2,k+1}(q, \varepsilon) e^{-qx} dq$$

$$(40) \quad S_3(x, \varepsilon) = \varepsilon^m \int_{3\varepsilon}^1 \frac{\tilde{R}(q/\varepsilon, q) e^{-qx}}{q(q+2\varepsilon)} dq$$

The middle integral in (37) has an expansion of the form (34) as it follows from the lemma below:

**Lemma 5.** *The terms  $S_1(x, \varepsilon)$  and  $S_2(x, \varepsilon)$  have expansions of the form (34), and  $S_3(x, \varepsilon)$  satisfies the estimates in (36). More precisely we have*

$$(41) \quad S_k(x; \varepsilon) = h_{k,1}(x) \varepsilon^{m-2} \ln \varepsilon + h_{k,2}(x) \varepsilon^{m-1} \ln \varepsilon + h_{k,3}(x) \varepsilon^{2m-4} (\ln \varepsilon)^2 + R_{k,0}(x; \varepsilon)$$

for  $k = 1, 2$ , where  $h_{k,j}$  satisfy the large  $x$  asymptotics

$$h_{1,1} \sim -\frac{(-2)^{m-2}}{\Gamma(m)} (1 + o(1)); \quad h_{1,2} \sim -\frac{(-2)^{m-2}(2x + o(x))}{\Gamma(m)}; \quad h_{1,3} \sim a_0 + o(1)$$

$$h_{2,1} = o(1); \quad h_{2,2} = O(1); \quad h_{2,3} = o(1)$$

where  $a_0$  is a constant, and  $R_{k,0}$  satisfy the estimates in (36).

*Proof.* We first show the result for  $S_1$  using (38). Since  $\tilde{H}_{1,k}$  are analytic in  $\varepsilon$ , we only need to analyze the integrals

$$(42) \quad \int_{3\varepsilon}^1 \frac{\ln^l(q/\varepsilon) e^{-qx}}{q+2\varepsilon} dq = - \int_1^\infty \frac{\ln^l(q/\varepsilon) e^{-qx}}{q+2\varepsilon} dq + \int_3^\infty \frac{\ln^l(\tau) e^{-\varepsilon\tau x}}{\tau+2} d\tau$$

where  $0 \leq l \leq 2$ . The first integral on the right hand side is a polynomial in  $\ln \varepsilon$  times a function analytic in  $\varepsilon$  with exponential decay in  $x$ , and the last one is equal to

$$(43) \quad \int_3^\infty \ln^l \tau (\tau^{-1} - 2\tau^{-2}) e^{-\varepsilon\tau x} d\tau + \int_3^\infty \frac{4 \ln^l \tau}{\tau^2(\tau+2)} e^{-\varepsilon\tau x} d\tau$$

To analyze the first term in (43) we need the following elementary result:

**Lemma 6.** *Assume  $l \geq 0$ ,  $x \geq x_+$  and  $|\varepsilon x| \leq 1$ . For  $n \geq 0$  we have*

$$(44) \quad \int_3^\infty e^{-\varepsilon\tau x} \tau^n (\ln \tau)^l d\tau = \frac{1}{(\varepsilon x)^{n+1}} \sum_{q=0}^l c_q^{(n;l)} \ln^q(\varepsilon x) + R_{a,n}(x, \varepsilon)$$

where  $c_q^{(n;l)}$  are constants with  $c_1^{(0;1)} = -1$ , and  $R_{a,n}$  satisfies

$$(45) \quad \left| \frac{\partial^k R_{a,n}(x, \varepsilon)}{\partial \varepsilon^k} \right| \lesssim x^k \quad (k \geq 0)$$

In addition, for  $n \leq -1$  we have

$$(46) \quad \int_3^\infty e^{-\varepsilon \tau x} \tau^n (\ln \tau)^l d\tau = (\varepsilon x)^{-1-n} \sum_{q=0}^{l+1} c_q^{(n;l)} \ln^q(\varepsilon x) + R_{a,n}(x, \varepsilon)$$

where  $c_1^{(-1;0)} = -1$ ,  $c_1^{(-2;0)} = 1$ , and  $R_{a,n}$  satisfies (45).

The proof of this Lemma is given in the Appendix.

Expansion of the first term in (43) follows directly from Lemma 6. The last term in (43) satisfies

$$(47) \quad \left| \frac{d^k}{d\varepsilon^k} \int_3^\infty \frac{4 \ln^l \tau}{\tau^2(\tau+2)} e^{-\varepsilon \tau x} d\tau \right| \lesssim \left| x^k \int_3^\infty \tau^{-3+k} \ln^l \tau e^{-\varepsilon \tau x} d\tau \right|$$

$$\lesssim \begin{cases} x^k, & 0 \leq k \leq 1; \\ x^2 |\varepsilon|^{2-k} (1 + |\ln^{l+1}(\varepsilon x)|), & 2 \leq k \leq m+1. \end{cases}$$

where the second integral in (47) is estimated using Lemma 6. Thus

$$\varepsilon^{(l+1)(m-2)+k} \int_3^\infty \frac{4 \ln^l \tau}{\tau^2(\tau+2)} e^{-\varepsilon \tau x} d\tau$$

satisfies (36) for all  $k \geq 0$  and  $l \geq 0$  by direct differentiation.

Thus combining (42) and (43) using Lemma 6 we see that

$$(48) \quad \varepsilon^{(l+1)(m-2)} \int_{3\varepsilon}^1 \frac{\ln^l(q/\varepsilon) e^{-qx}}{q+2\varepsilon} dq = \varepsilon^{(l+1)(m-2)} \ln^l \varepsilon \int_1^\infty \frac{e^{-qx}}{q+2\varepsilon} dq$$

$$+ \sum_{n=-2}^{-1} (-2)^{-1-n} \varepsilon^{(l+1)(m-2)-1-n} x^{-1-n} \sum_{q=0}^{l+1} c_q^{(n;l)} \ln^q(\varepsilon x) + R_{0,0}(x, \varepsilon)$$

where  $R_{0,0}$  satisfies (36).

Letting  $l = 0, 1, 2$  in (48) we have

$$(49) \quad \varepsilon^{m-2} \tilde{H}_{1,k}(\varepsilon) \int_{3\varepsilon}^1 \frac{e^{-qx}}{q+2\varepsilon} dq = h_{a,1}(x) \varepsilon^{m-2} \ln \varepsilon + h_{a,2}(x) \varepsilon^{m-1} \ln \varepsilon + R_{0,1}(x, \varepsilon)$$

where  $h_{a,1}(x) = -1 + o(1)$  and  $h_{a,2}(x) = -2x(1 + o(1))$  for large  $x$ ,

$$(50) \quad \varepsilon^{2m-4} \tilde{H}_{2,k}(\varepsilon) \int_{3\varepsilon}^1 \frac{\ln(q/\varepsilon) e^{-qx}}{q+2\varepsilon} dq = c_2^{(-1;1)} \varepsilon^{2m-4} \ln^2 \varepsilon + h_{b,1}(x) \varepsilon^{2m-4} \ln \varepsilon + R_{0,2}(x, \varepsilon)$$

where  $h_{b,1}(x) = O(1)$  for large  $x$ , and

$$(51) \quad \varepsilon^{3m-6} \tilde{H}_{3,k}(\varepsilon) \int_{3\varepsilon}^1 \frac{\ln^2(q/\varepsilon) e^{-qx}}{q+2\varepsilon} dq = R_{0,3}(x, \varepsilon)$$

where  $R_{0,i}$  ( $0 \leq i \leq 3$ ) satisfies (36). Thus (41) follows from (49), (50), and (51). Note that

$$\tilde{H}_{1,1}(0) = \frac{(-2)^{m-2}}{\Gamma(m)} \text{ by Corollary 3.}$$

To show (41) for  $S_2$  we write  $\ln(q/\varepsilon) = \ln q - \ln \varepsilon$  in (39). By Corollary 3

$$\int_{2\varepsilon}^1 \tilde{H}_{2,n}(q, \varepsilon) e^{-qx} dq$$

is entire in  $\varepsilon$  with its  $k$ -th derivative in  $\varepsilon$  bounded by  $\text{const.} x^{k-1}$ . Thus the term containing  $\tilde{H}_{2,1}$  satisfies (36) and the terms containing  $\ln \varepsilon \tilde{H}_{2,2}$  and  $\ln^2 \varepsilon \tilde{H}_{2,3}$  satisfy (41).



The term with  $\ln q \tilde{H}_{2,2}$  is analyzed using integration by parts:

$$(52) \quad \varepsilon^{m-2} \int_{3\varepsilon}^1 \ln q \tilde{H}_{2,2}(q, \varepsilon) e^{-qx} dq = \varepsilon^{m-2} \tilde{H}_{2,2}(q, \varepsilon) e^{-qx} (q \ln q - q) \Big|_{3\varepsilon}^1 \\ - \varepsilon^{m-2} \int_{3\varepsilon}^1 (q \ln q - q) \frac{\partial(\tilde{H}_{2,2}(q, \varepsilon) e^{-qx})}{\partial q} dq$$

where the last term satisfies (36) by direct calculation and counting powers of  $\varepsilon$ .

The term containing  $\ln q \tilde{H}_{2,3}(q, \varepsilon)$  can be similarly analyzed using integration by parts, which gives

$$\varepsilon^{2m-4} \ln \varepsilon \int_{3\varepsilon}^1 \ln q \tilde{H}_{2,2}(q, \varepsilon) e^{-qx} dq = -\varepsilon^{2m-4} \ln \varepsilon \tilde{H}_{2,3}(1, \varepsilon) e^{-x} + R_s(x, \varepsilon)$$

where  $R_s$  satisfies (36). The term containing  $(\ln q)^2 \tilde{H}_{2,3}(q, \varepsilon)$  satisfies (36) by direct calculation and counting powers of  $\varepsilon$ .

Finally we show that  $S_3(x, \varepsilon)$  satisfies the estimate (36). Denoting  $\partial_{(i,j)} \tilde{R}(u, v) = \partial^{i+j} \tilde{R}(u, v) / (\partial u)^i (\partial v)^j$ , we have by Lemma 2

$$(53) \quad \left| \frac{\partial^k}{\partial \varepsilon^k} \left( \varepsilon \int_{3\varepsilon}^1 \frac{\tilde{R}(q/\varepsilon, q) e^{-qx}}{q(q+2\varepsilon)} dq \right) \right| = \left| \frac{\partial^k}{\partial \varepsilon^k} \int_3^{1/\varepsilon} \frac{\tilde{R}(\tau, \varepsilon \tau) e^{-\varepsilon \tau x}}{\tau(\tau+2)} d\tau \right| \\ \lesssim \sum_{i+j=k} \left| \int_3^{1/\varepsilon} \frac{\partial_{(0,i)} \tilde{R}(\tau, \varepsilon \tau) \tau^{i-1} x^j e^{-\varepsilon \tau x}}{\tau+2} d\tau \right| + \sum_{i+j=k-1} \left| \frac{\partial_{(i,0)} \tilde{R}(1/\varepsilon, 1) e^{-x}}{\varepsilon^{2i+j}} \right| \\ \lesssim (|\ln \varepsilon| + 1)^4 \left( |\varepsilon^{1-k}| e^{-x} + \sum_{i+j=k} |\varepsilon^{1-i} x^j| \right)$$

for  $0 \leq k \leq m+1$ . Thus

$$\left| \frac{\partial^n S_3(x, \varepsilon)}{\partial \varepsilon^n} \right| \lesssim \sum_{j+k=n} \left| \varepsilon^{m-1-j} \frac{\partial^k}{\partial \varepsilon^k} \left( \varepsilon \int_{3\varepsilon}^1 \frac{\tilde{R}(q/\varepsilon, q) e^{-qx}}{q(q+2\varepsilon)} dq \right) \right| \lesssim (|\ln \varepsilon| + 1)^4 \varepsilon^{m-n}$$

□

Finally  $s'$  and  $s''$  are similarly analyzed, finishing the proof of 4. □

**3.2. Detailed behavior of Jost functions.** Recall that  $y(x) = e^{-\varepsilon x}(1 + s(x))$  satisfies (3). For small  $\varepsilon$ , the leading order of (3) is

$$(54) \quad f''(x) = v_1 x^{-m} f(x)$$

The solutions to (54) are modified Bessel functions  $\Phi_{1,2}$  with  $\Phi_1(x) = 1 + o(1)$  and  $\Phi_2(x) = x(1 + o(1))$  for large  $x$ .

For convenience we take  $v_1=1$ , then

$$\Phi_1(x) = (m-2)^{1/(m-2)} \Gamma\left(\frac{m-1}{m-2}\right) \sqrt{x} I_{\frac{1}{m-2}}\left(\frac{2x^{1-m/2}}{m-2}\right) \\ \Phi_2(x) = \frac{(m-2)^{-1/(m-2)}}{\Gamma\left(\frac{1}{m-2}\right)} \sqrt{x} K_{\frac{1}{m-2}}\left(\frac{2x^{1-m/2}}{m-2}\right)$$

where  $I_n$  and  $K_n$  denote the modified Bessel functions of the first and second kind respectively.

**Proposition 7.** For arbitrarily small  $\delta > 0$ ,  $x \geq x_+$  and  $|\varepsilon| \leq 1/x$  we have

$$(55) \quad y_+(x; \varepsilon) = r(\varepsilon) \left( \Phi_1(x) + B_1(x) \varepsilon^{m-1} \ln \varepsilon + \hat{f}_a(x, \varepsilon) \right)$$

where

$$(56) \quad r(\varepsilon) := (1 + a_1 \varepsilon^{m-2} \ln \varepsilon + a_0 \varepsilon^{2m-4} (\ln \varepsilon)^2)$$

where  $a_{0,1}$  are constants,  $B_1(x)$  is a linear combination of  $\Phi_{1,2}(x)$  with  $B_2(x) \sim \text{const.}x$ ,  $\hat{f}_a(x, \varepsilon) \lesssim |\varepsilon x|$ , and

$$(57) \quad \sup_{1 \leq k \leq m-1} \left| x^{-k} \frac{\partial^k \hat{f}_a(x; \varepsilon)}{\partial \varepsilon^k} \right| < \infty; \quad \sup_{m \leq k \leq m+1} \left| \varepsilon^{k-1-m+\delta} x^{-m-1} \frac{\partial^k \hat{f}_a(x; \varepsilon)}{\partial \varepsilon^k} \right| < \infty$$

Furthermore, the expansion (55) is differentiable in  $x$ , i.e. (57) holds with  $\hat{f}_a(x; \varepsilon)$  replaced by  $x \hat{f}'_a(x; \varepsilon)$ .

*Proof.* It follows from Lemma 4 that  $y = e^{-\varepsilon x}(1+s)$  has the following expansion

$$(58) \quad y(x; \varepsilon) = h_0(x) + h_1(x)\varepsilon + h_2(x)\varepsilon^{m-2} \ln \varepsilon + h_3(x)\varepsilon^{m-1} \ln \varepsilon + h_4(x)\varepsilon^{2m-4}(\ln \varepsilon)^2 + \tilde{H}(x; \varepsilon)$$

where  $h_0 \sim 1$ ,  $h_2 \sim a_1$ ,  $h_4 \sim a_0$  for large  $x$ , and  $\tilde{H}(x; \varepsilon)$  satisfies (57) with  $|\tilde{H}(x; \varepsilon)| \lesssim |\varepsilon^2 x^2|$ . Plugging this expansion back into (3) (recall that the asymptotics (58) is differentiable by Lemma 4) we have

$$(59) \quad 0 = \left( h_0''(x) - \frac{1}{x^m} h_0(x) \right) + \left( h_1''(x) - \frac{1}{x^m} h_1(x) \right) \varepsilon + \left( h_2''(x) - \frac{1}{x^m} h_2(x) \right) \varepsilon^{m-2} \ln \varepsilon + \dots$$

Standard asymptotic arguments for  $\varepsilon \rightarrow 0$  show that all the coefficients above must be zero, and thus  $h_i$  satisfies (54), implying  $h_0(x) = \Phi_1(x)$ , and  $h_2(x) = a_1 \Phi_1(x)$ ,  $h_4(x) = a_0 \Phi_1(x)$  since  $h_0 \sim 1$ ,  $h_2 \sim a_1$ , and  $h_4 \sim a_0$ .

Thus dividing (58) by (56) we obtain (55) for some  $B_1$ . Substituting (55) into (3) and examining the coefficients of  $\varepsilon^{m-1} \ln \varepsilon$  we see that  $B_1$  satisfies (54) and is thus a linear combination of  $\Phi_{1,2}$ . The large  $x$  behavior of  $B_1$  follows from (34) and (35). Differentiability of (55) follows from the last paragraph of Lemma 4.  $\square$

**Note 4.** (i) By symmetry a similar conclusion holds for the Jost solution decaying as  $x \rightarrow -\infty$ .

(ii) Without loss of generality we can assume  $r(\varepsilon) \neq 0$ , since otherwise we can modify the definition of  $r(\varepsilon)$  by adding  $\varepsilon$  to it.

**3.3. Estimating the Jost functions for large  $\varepsilon$ .** In Section 4 we need the behavior of  $\partial^n s(x; \varepsilon)/\partial \varepsilon^n$  for  $\varepsilon$  in  $\mathbb{H}$ . Let  $\mathcal{D} = (x_+, \infty) \times \mathbb{H}$ . First we prove the following lemma:

**Lemma 8.** Assume  $s(x; \varepsilon)$  solves (6) with  $|V^{(k)}(x)| \lesssim x^{-m-k}$  for  $x \geq x_+$  and  $0 \leq k \leq m+2$ , and  $s(x; \varepsilon) = o(1)$  for  $x \rightarrow \infty$ . Then for arbitrary fixed  $x_1 \in \mathbb{R}$  the function  $s(x; \cdot)$  is analytic in  $\mathbb{H}$  and continuous in  $\mathbb{H}$  for  $x \geq x_1$ . Moreover, we have

$$(60) \quad |s(x; \varepsilon)| \lesssim (|\varepsilon| \langle x \rangle + 1)^{-1} \langle x \rangle^{-m+2}; \quad |s'(x; \varepsilon)| \lesssim (|\varepsilon| \langle x \rangle + 1)^{-1} \langle x \rangle^{-m+1}$$

uniformly in  $\mathcal{D}$ , and for  $|\varepsilon| \geq 1/\langle x \rangle$  we have

$$(61) \quad \left| \frac{\partial^n s(x; \varepsilon)}{\partial \varepsilon^n} \right| \lesssim |\varepsilon|^{-1-n} \langle x \rangle^{1-m}; \quad \left| \frac{\partial^n s'(x; \varepsilon)}{\partial \varepsilon^n} \right| \lesssim |\varepsilon|^{-1-n} \langle x \rangle^{-m}; \quad 0 \leq n \leq m+1$$

A similar conclusion is true for the other Jost solution, i.e. (60) and (61) holds for both  $s_+$  and  $s_-$ .

*Proof.* We write (6) in integral form:

$$(62) \quad s(x) = \int_{-\infty}^x \int_{-\infty}^t e^{-2\varepsilon(t'-t)} V(t') dt' dt + \int_{-\infty}^x \int_{-\infty}^t e^{-2\varepsilon(t'-t)} V(t') s(t') dt' dt := T_1(x; V) + L s$$

It is straightforward to check that  $|T_1(x; V)| \lesssim x^{2-m}$ , and by integration by parts,

$$(63) \quad |T_1(x; V)| \leq \frac{1}{2|\varepsilon|} \left| \int_{-\infty}^x \left( V(t) - \int_{-\infty}^t e^{-2\varepsilon(t'-t)} V'(t') dt' \right) dt \right| \lesssim |\varepsilon|^{-1} x^{1-m}$$

Thus

$$(64) \quad |T_1(x; V)| \lesssim \min(x^{2-m}, |\varepsilon|^{-1} x^{1-m}) \lesssim (|\varepsilon x| + 1)^{-1} x^{2-m}$$

uniformly in  $x \geq x_+$  and  $\varepsilon \in \overline{\mathbb{H}}$ . We analyze (62) in the Banach space  $\mathcal{B}$  of functions  $f : \mathcal{D} \rightarrow \mathbb{C}$ , such that  $f(x, \cdot)$  is analytic in  $\mathbb{H}$ , continuous in  $\overline{\mathbb{H}}$ , with the norm

$$(65) \quad \|s\| = \sup_{(x, \varepsilon) \in \mathcal{D}} |s(x, \varepsilon)| < \infty$$

We see that  $T_1 \in \mathcal{B}$  and

$$(66) \quad \|L\| \leq \left| \int_{\infty}^x \int_{\infty}^t V(t') dt' dt \right| \leq \text{const.} x^{2-m}$$

Thus if  $x_0$  is sufficiently large then (62) is contractive in  $\mathcal{B}$  and has a unique solution for  $x \geq x_0$ . Then, the estimate (60) is obtained by taking  $x_0$  sufficiently large and writing

$$(67) \quad |s(x, \varepsilon)| \leq |(1 - L)^{-1} T_1|$$

For the derivatives, the proof is by induction on  $n$ . The equation for  $u = \frac{d}{d\varepsilon} s(x, \varepsilon)$  is

$$(68) \quad u'' - 2\varepsilon u' - V(x)u = 2s'(x)$$

We see, by direct differentiation of the rhs of (62) that

$$(69) \quad s'(x; \varepsilon) = \int_{\infty}^x e^{-2\varepsilon(t'-x)} V(t') dt' + \int_{\infty}^x e^{-2\varepsilon(t'-x)} V(t') s(t') dt'$$

Thus

$$(70) \quad \left| s'(x, \varepsilon) + \frac{1}{2\varepsilon} \left( V(x) - \int_{\infty}^x e^{-2\varepsilon(t'-x)} V'(t') dt' \right) \right| \lesssim |\varepsilon|^{-1} x^{2(1-m)}$$

implying

$$(71) \quad |s'(x, \varepsilon)| \lesssim |\varepsilon|^{-1} x^{-m}$$

Also combining (70) with (6) we have

$$(72) \quad |s''(x, \varepsilon)| \leq |2\varepsilon s'(x, \varepsilon) + V(x) + V(x)s(x, \varepsilon)| \lesssim |\varepsilon|^{-1} x^{-m-1}$$

By (68) we have

$$(73) \quad u(x) = T_1(x; 2s') + L u$$

(see (62)). Using (71) and (72) we obtain

$$(74) \quad |T_1(x; 2s')| \leq \frac{1}{|\varepsilon|} \left| \int_{\infty}^x \left( s'(t) - \int_{\infty}^t e^{-2\varepsilon(t'-t)} s''(t') dt' \right) dt \right| \lesssim |\varepsilon|^{-2} x^{1-m}$$

Thus (66) and (73) imply

$$|u(x, \varepsilon)| \lesssim |\varepsilon|^{-2} x^{1-m}$$

$$|u'(x, \varepsilon)| = \left| 2 \int_{\infty}^x e^{-2\varepsilon(t'-x)} s'(t') dt' + \int_{\infty}^x e^{-2\varepsilon(t'-x)} V(t') u(t') dt' \right| \lesssim |\varepsilon|^{-2} x^{-m}$$

Taking  $k$   $\varepsilon$ -derivatives of (68) and letting  $u_k(x, \varepsilon) = \partial^k s(x, \varepsilon) / \partial \varepsilon^k$  we have

$$(75) \quad u_k'' - 2\varepsilon u_k' - V(x)u_k = 2k u_{k-1}'$$

which gives by induction

$$|u_k(x, \varepsilon)| \leq |(1 - L)^{-1} T_1(x; 2k u_{k-1}')| \lesssim |\varepsilon|^{-1-k} |x|^{1-m}$$

$$|u_k'(x, \varepsilon)| \lesssim |\varepsilon|^{-1-k} |x|^{-m}$$

Finally, for  $x_1 \leq x \leq x_0$  existence and analyticity of the solution follow from standard analytic dependence on parameters. Thus (60) and (61) only need to be verified for large  $\varepsilon$  since the  $x$  dependence is irrelevant. Note that

$$(76) \quad s(x) = s(x_0) + s'(x_0) + \int_{x_0}^x \int_{x_0}^t e^{-2\varepsilon(t'-t)} V(t') dt' dt$$

$$+ \int_{x_0}^x \int_{x_0}^t e^{-2\varepsilon(t'-t)} V(t') s(t') dt' dt := T_{2,0}(x) + L_0 s$$

Let  $\|s\|_1 = \sup_{x_1 \leq \tau \leq x_0} |s(\tau)e^{-\mu(\tau-x_1)}|$  with suitably large  $\mu$ . Under this norm (76) is contractive (with contractivity factor  $O(\mu^{-2})$ ) which gives  $|s(x)| \lesssim 1/|\varepsilon|$ . Note that

$$(77) \quad \left| \int_{x_0}^x \int_{x_0}^t e^{-2\varepsilon(t'-t)} V(t') dt' dt \right| = \left| \frac{1}{2\varepsilon} \int_{x_0}^x \left( e^{-2\varepsilon(x_0-t)} V(x_0) - V(t) + \int_{x_0}^t e^{-2\varepsilon(t'-t)} V'(t') dt' \right) dt \right| \lesssim 1/|\varepsilon|$$

Similarly using (75) we see that  $|\partial^n s(x)/\partial \varepsilon^n| \lesssim 1/|\varepsilon|^{n+1}$  by induction.  $\square$

**Corollary 9.** *For  $\pm(x-x_+) \geq 0$  and  $|\varepsilon| \leq 1$ , the Jost solution  $y_\pm$  in (5) satisfies*

$$(78) \quad y_\pm(x) = r_\pm(\varepsilon) e^{\mp \varepsilon x} \left( \Phi_1^\pm(x) + \frac{B_2^\pm(x) \varepsilon^{m-1} \ln \varepsilon}{1 + \varepsilon \langle x \rangle} + R_f^\pm(x; \varepsilon) \right)$$

where  $r_\pm(\varepsilon)$  are of the form (56), the modified Bessel functions  $\tilde{\Phi}_1^\pm(x) \sim 1$  and  $B_2^\pm(x) \sim \text{const.} x$  for  $x \rightarrow \pm\infty$ , and (57) holds with  $R_f^\pm(x; \varepsilon)$  replacing  $\hat{f}_a$ . Also

$$(79) \quad y'_\pm(x) = r_\pm(\varepsilon) e^{\mp \varepsilon x} \left( \Phi_1^{\prime \pm}(x) + \frac{B_2^{\prime \pm}(x) \varepsilon^{m-1} \ln \varepsilon}{1 + \varepsilon \langle x \rangle} + r_f^{\prime \pm}(x; \varepsilon) \right)$$

where (57) holds with  $\langle x \rangle r_f^{\prime \pm}(x)$  replacing  $\hat{f}_a$ .

*Proof.* Without loss of generality we show the result for  $y_+$ . By Proposition 7 we see that (78) holds for  $|\varepsilon| \leq 1/x$ , since multiplying by  $e^{\varepsilon x}$  and  $1/(1 + \varepsilon \langle x \rangle)$  does not change the structure (55). For  $1/x \leq |\varepsilon| \leq 1$ , we only need to show (57) is valid with  $\hat{f}_a$  replaced by  $R_f(x; \varepsilon)$ . By (61) we have for  $1/x \leq |\varepsilon| \leq 1$

$$(80) \quad \left| \frac{\partial^n s_+(x; \varepsilon)}{\partial \varepsilon^n} \right| \lesssim x^{n+2-m}; \quad 0 \leq n \leq m+1$$

Now  $(r_+(\varepsilon))^{-1}(1 + s_+(x; \varepsilon))$  satisfies the estimates for  $\hat{f}_a$  in (57) by (80) (cf. Note 4). Since  $\frac{B_2^+(x) \varepsilon^{m-1} \ln \varepsilon}{1 + \varepsilon \langle x \rangle}$  obviously satisfies the estimates in (57), we see that  $R_f$  satisfies (57). Differentiability of (78) follows from Lemma 4 and Lemma 8.  $\square$

#### 4. PROOF OF THEOREM 1

We will show that the leading behavior of the time decay is of the form  $\hat{r}_0(x)/t^m$  with the bounds for the remainder specified in Theorem 1.

**4.1. Jost functions on the real line.** Assume  $V(x) = v_1 x^{-m}$  for  $x \geq x_+$  and  $V(x) = v_2 x^{-m}$  for  $x \leq x_-$ . Let  $y_\pm$  be the two linearly independent Jost functions (solutions of (3)) with  $y_+ \sim e^{-\varepsilon x}$  for  $x \gg 1$  and  $\varepsilon \in \mathbb{H}$ , and  $y_- \sim e^{\varepsilon x}$  for  $x \ll -1$  and  $\varepsilon \in \mathbb{H}$ . We now show the following

**Proposition 10.** *The Jost function  $y_+$  is analytic in  $\varepsilon \in \mathbb{H}$  for all  $x \in \mathbb{R}$ . Moreover, for  $|\varepsilon| \leq 1$  it satisfies*

$$(81) \quad y_\pm(x) = r_\pm(\varepsilon) e^{\mp \varepsilon x} \left( r_0^\pm(x) + \frac{r_1^\pm(x) \varepsilon^{m-1} \ln \varepsilon}{1 + \varepsilon \langle x \rangle} + R_0^\pm(x; \varepsilon) \right)$$

where  $|r_0^\pm(x)| \lesssim \langle x \rangle$  and  $|r_1^\pm(x)| \lesssim \langle x \rangle^2$  for all  $x \in \mathbb{R}$ , and for  $x \geq x_\pm$ ,  $r_0^\pm(x)$  and  $r_1^\pm(x)$  are explicit modified Bessel functions with  $|r_0^\pm(x)| \lesssim 1$ ,  $|r_1^\pm(x)| \lesssim \langle x \rangle$  (cf. Corollary 9). Furthermore  $R_0^\pm(x; 0) = 0$  and for arbitrarily small  $\delta > 0$  we have

$$(82) \quad \sup_{\substack{0 \leq k \leq m-1 \\ \pm(x-x_\mp) \geq 0}} \left| \langle x \rangle^{-k} \frac{\partial^k R_0^\pm(x; \varepsilon)}{\partial \varepsilon^k} \right| \lesssim 1; \quad \sup_{\substack{m \leq k \leq m+1 \\ \pm(x-x_\mp) \geq 0}} \left| |\varepsilon|^{k-m+\delta} \langle x \rangle^{-m} \frac{\partial^k R_0^\pm(x; \varepsilon)}{\partial \varepsilon^k} \right| \lesssim 1$$

$$\sup_{\substack{0 \leq k \leq m-1 \\ \pm(x-x_\mp) \leq 0}} \left| \langle x \rangle^{-k-1} \frac{\partial^k R_0^\pm(x; \varepsilon)}{\partial \varepsilon^k} \right| \lesssim 1; \quad \sup_{\substack{m \leq k \leq m+1 \\ \pm(x-x_\mp) \leq 0}} \left| |\varepsilon|^{k-m+\delta} \langle x \rangle^{-m-1} \frac{\partial^k R_0^\pm(x; \varepsilon)}{\partial \varepsilon^k} \right| \lesssim 1$$

In addition, (81) is differentiable in  $x$ , i.e. (82) holds with  $R_0^\pm(x; \varepsilon)$  replaced with  $\langle x \rangle R_0'^\pm(x; \varepsilon)$ . For  $|\varepsilon| \geq 1$  we have

$$(83) \quad y_\pm(x; \varepsilon) = e^{\mp \varepsilon x} (1 + s_\pm(x; \varepsilon))$$

where  $s_\pm(x; \varepsilon)$  satisfies (61) for  $\pm(x - x_\mp) \geq 0$ , and

$$(84) \quad \left| \frac{\partial^n s_\pm(x; \varepsilon)}{\partial \varepsilon^n} \right| \lesssim |\varepsilon|^{-1} \langle x \rangle^n; \quad \left| \frac{\partial^n s'_\pm(x; \varepsilon)}{\partial \varepsilon^n} \right| \lesssim \langle x \rangle^n; \quad 0 \leq n \leq m+1$$

for  $\pm(x - x_\mp) \leq 0$ .

*Proof.* We only analyze  $y_+$  since  $y_-$  is similar. For  $x \geq x_+$ , the behavior of  $y_+$  for small  $\varepsilon$  is given in Corollary 9 and the behavior for large  $\varepsilon$  is given in Lemma 8.

First assume  $|\varepsilon| \leq 1$ . In the middle region  $x_- \leq x \leq x_+$  there exist two linearly independent solutions analytic in  $\varepsilon$  (by analytic dependence on parameters of ODE) and the  $x$  bounds are irrelevant. Clearly  $y_+$  is a linear combination of these two solutions, and thus the analytic structure of  $y_+$  is preserved (cf. Corollary 9). For  $x \leq x_-$ , by (60) we can assume  $x_-$  is large enough such that  $y_-(x) \neq 0$ . By standard ODE results  $y_+$  satisfies

$$(85) \quad y_+(x; \varepsilon) = y_-(x; \varepsilon) \left( \frac{y_+(x_-; \varepsilon)}{y_-(x_-; \varepsilon)} - W(\varepsilon) \int_{x_-}^x \frac{1}{y_-^2(t; \varepsilon)} dt \right)$$

where  $W(\varepsilon) = y_+ y'_- - y'_+ y_-$  is the Wronskian.

One can verify by direct calculation that  $y_+$  solves (3) for  $x \leq x_-$  and is differentiable at  $x_-$ . By Corollary 9 and the reasoning above we see that for  $x_- \leq x \leq x_+$

$$(86) \quad y_+(x; \varepsilon) = r(\varepsilon) e^{-\varepsilon x} \left( \Phi_0(x) + \frac{\hat{B}_2(x) \varepsilon^{m-1} \ln \varepsilon}{1 + \varepsilon \langle x \rangle} + \hat{R}_f(x; \varepsilon) \right)$$

where  $\Phi_0$  and  $\hat{B}_2$  are smooth, and with (57) holds with  $\hat{R}_f$  replacing  $\hat{f}_a$ . Also note that (78) holds for  $x \leq x_-$ .

**Note 5.** Assume  $|\varepsilon| \leq 1$ . Since the Wronskian  $W$  is independent of  $x$ , by direct calculation using (86) and (78) at  $x_-$  for  $|\varepsilon| \leq 1$  we have

$$(87) \quad W(\varepsilon) = r_+(\varepsilon) r_-(\varepsilon) (q_1 \varepsilon^{m-1} \ln \varepsilon + q_2(\varepsilon))$$

where  $q_1$  is a constant, and by Corollary 9 we have  $\frac{d^k q_2(\varepsilon)}{d\varepsilon^k}$  is bounded for  $0 \leq k \leq m-1$ , and  $\varepsilon^{k-m+\delta} \frac{d^k q_2(\varepsilon)}{d\varepsilon^k}$  is bounded for  $m \leq k \leq m+1$  and arbitrarily small  $\delta > 0$ . In the absence of zero-energy resonance,  $y_+$  and  $y_-$  are linearly independent for small  $\varepsilon$ , and thus  $q_2(0) = W(0) \neq 0$ .

Assume  $|\varepsilon| \geq 1$ . By direct calculation using (83) at  $x_-$  for  $|\varepsilon| \geq 1$ , denoting

$$(88) \quad q_3(\varepsilon) = W(\varepsilon) - 2\varepsilon$$

we have  $\frac{d^k q_3(\varepsilon)}{d\varepsilon^k}$  is bounded for  $0 \leq k \leq m+1$ .

By (85) we have

$$e^{\varepsilon x} y_+(x; \varepsilon) = r(\varepsilon) \left( \hat{\Phi}_1(x) + \frac{B_3(x) \varepsilon^{m-1} \ln \varepsilon}{1 + \varepsilon \langle x \rangle} + x \hat{f}_b(x; \varepsilon) \right)$$

where  $\hat{\Phi}_1(x) = \Phi_1^-(x) \left( \frac{\Phi_1^+(x_-)}{\Phi_1^-(x_-)} - W(0) \int_{x_-}^x \frac{1}{\Phi_1^{-2}(t)} dt \right)$  satisfies  $|\hat{\Phi}_1(x)| \lesssim \langle x \rangle$ ,  $B_3$  is a smooth function with  $|B_3| \lesssim \langle x \rangle^2$  and (57) holds with  $\hat{f}_a$  replaced by  $\hat{f}_b$  by (85), which can be checked using the fact that for any bounded function  $f$  we have

$$\left| e^{2\varepsilon x} \int_{x_-}^x e^{-2\varepsilon t} t^k f(t) dt \right| \lesssim |x|^{k+1}$$

Assume  $|\varepsilon| \geq 1$ . In the middle region  $x_- \leq x \leq x_+$  the result (83) follows from Lemma 8.

For  $x \leq x_-$ , we denote  $f_{\pm} = 1 + s_{\pm}$  and by (85) and Lemma 8 we have

$$(89) \quad e^{\varepsilon x} y_+(x; \varepsilon) = f_-(x; \varepsilon)(1 + s_3(x; \varepsilon))$$

where

$$(90) \quad s_3(x; \varepsilon) = -1 + \frac{f_+(x_-; \varepsilon)}{f_-(x_-; \varepsilon)} e^{2\varepsilon(x-x_-)} - (2\varepsilon + q_3(\varepsilon)) \int_{x_-}^x \frac{e^{2\varepsilon(x-t)}}{f_-^2(t; \varepsilon)} dt$$

$$= \frac{s_+(x_-; \varepsilon) - s_-(x_-; \varepsilon)}{f_-(x_-; \varepsilon)} e^{2\varepsilon(x-x_-)} - q_3(\varepsilon) \int_{x_-}^x \frac{e^{2\varepsilon(x-t)}}{f_-^2(t; \varepsilon)} dt + 2\varepsilon \int_{x_-}^x \frac{e^{2\varepsilon(x-t)} s_-(t; \varepsilon) (s_-(t; \varepsilon) + 2)}{f_-^2(t; \varepsilon)} dt$$

Using (61) in Lemma 8 we see that the first term in (90) satisfies the estimates in (84). By integration by parts we obtain

$$(91) \quad q_3(\varepsilon) \int_{x_-}^x \frac{e^{2\varepsilon(x-t)}}{f_-^2(t; \varepsilon)} dt = \frac{q_3(\varepsilon)}{2\varepsilon} \left( \frac{e^{2\varepsilon(x-x_-)}}{f_-^2(x_-; \varepsilon)} - \frac{1}{f_-^2(x; \varepsilon)} + \int_{x_-}^x e^{2\varepsilon(x-t)} \left( \frac{1}{f_-^2(t; \varepsilon)} \right)' dt \right)$$

and

$$(92) \quad 2\varepsilon \int_{x_-}^x \frac{e^{2\varepsilon(x-t)} s_-(t; \varepsilon) (s_-(t; \varepsilon) + 2)}{f_-^2(t; \varepsilon)} dt = \frac{e^{2\varepsilon(x-x_-)} s_-(x_-; \varepsilon) (s_-(x_-; \varepsilon) + 2)}{f_-^2(x_-; \varepsilon)}$$

$$- \frac{s_-(x; \varepsilon) (s_-(x; \varepsilon) + 2)}{f_-^2(x; \varepsilon)} + \int_{x_-}^x e^{2\varepsilon(x-t)} \left( \frac{s_-(t; \varepsilon) (s_-(t; \varepsilon) + 2)}{f_-^2(t; \varepsilon)} \right)' dt$$

Thus using the estimates in Lemma 8 as well as (90) we see that (84) holds with  $s_+$  replaced by  $s_3$ . The conclusion then follows, since by (83) and (89) we have  $s_+(x; \varepsilon) = s_-(x; \varepsilon) + s_3(x; \varepsilon) - s_-(x; \varepsilon) s_3(x; \varepsilon)$  where  $s_-$  satisfies (61).  $\square$

**4.2. Analysis of  $\hat{\psi}$ .** The solution of the Laplace transformed equation is

$$(93) \quad \hat{\psi}(x, \varepsilon) = \mathcal{G}(\psi_1 + \varepsilon \psi_0); \quad \mathcal{G}(\psi) := \frac{1}{W(\varepsilon)} \left( y_-(x) \int_{\infty}^x y_+(u) \psi(u) du - y_+(x) \int_{-\infty}^x y_-(u) \psi(u) du \right)$$

Note that  $W(\varepsilon) \neq 0$  for  $\varepsilon \in \overline{\mathbb{H}}$  by the assumption of no bound state and no zero energy resonance.

**Proposition 11.** For  $\varepsilon \in \overline{\mathbb{H}} \setminus \{0\}$  we have

$$(94) \quad \mathcal{G}(\psi_1) = r_3(x) \frac{\varepsilon^{m-1} \ln \varepsilon}{(1 + \varepsilon \langle x \rangle)^{m+2}} + \mathcal{G}_0(\psi_1(x)) + R(x; \varepsilon)$$

where

$$(95) \quad \|\langle x \rangle^{-2} r_3(x)\|_{\infty} \lesssim \|\langle x \rangle^2 \psi_1(x)\|_1$$

$$(96) \quad \mathcal{G}_0(\psi) := \frac{1}{2(1 + \varepsilon)} \left( \int_{\infty}^x e^{-\varepsilon(u-x)} \psi(u) du - \int_{-\infty}^x e^{\varepsilon(u-x)} \psi(u) du \right)$$

$$(97) \quad \|\langle x \rangle^{-k-2} \frac{\partial^k}{\partial \varepsilon^k} R(x, \varepsilon)\|_{\infty} \lesssim (1 + \varepsilon)^{-2} \|\langle x \rangle^{k+2} \psi_1(x)\|_1 \quad (0 \leq k \leq m-1)$$

$$\|\langle x \rangle^{-m-2} \frac{\partial^m}{\partial \varepsilon^m} R(x, \varepsilon)\|_{\infty} \lesssim (|\varepsilon|^{\delta} + |\varepsilon|)^{-2} \|\langle x \rangle^{m+2} \psi_1(x)\|_1$$

for arbitrary  $\delta > 0$ .

*Proof.* The fact that  $\mathcal{G}(\psi_1)$  is analytic in  $\varepsilon \in \mathbb{H}$  follows easily from Proposition 10.

We first consider the case  $|\varepsilon| \leq 1$ . By (81) we have

$$(98) \quad y_-(x) \int_{\infty}^x y_+(u) \psi_1(u) du - y_+(x) \int_{-\infty}^x y_-(u) \psi_1(u) du = r_+(\varepsilon) r_-(\varepsilon) \sum_{k,j=1,2} \tilde{G}_{j,k}(x, \varepsilon)$$

where

$$(99) \quad \tilde{G}_{j,k}(x, \varepsilon) = e^{\varepsilon x} G_j^-(x; \varepsilon) \int_{-\infty}^x e^{-\varepsilon u} G_k^+(u; \varepsilon) \psi_1(u) du - e^{-\varepsilon x} G_j^+(x; \varepsilon) \int_{-\infty}^x e^{\varepsilon u} G_k^-(u; \varepsilon) \psi_1(u) du$$

$G_1^\pm(x; \varepsilon) = \frac{r_1^\pm(x) \varepsilon^{m-1} \ln \varepsilon}{1 + \varepsilon \langle x \rangle}$ , and  $G_2^\pm(x; \varepsilon) = r_0^\pm(x) + R_0^\pm(x; \varepsilon)$ . By Proposition 10 clearly (82) holds with  $G_2^\pm$  instead of  $R_0^\pm$ . We denote for  $(j, k) = (0, 1)$  or  $(1, 0)$

$$\hat{G}_{j,k}(x) = \left( r_j^-(x) \int_{-\infty}^x r_k^+(u) \psi_1(u) du - r_j^+(x) \int_{-\infty}^x r_k^-(u) \psi_1(u) du \right)$$

By direct calculation for  $0 \leq k \leq m-1$

$$(100) \quad \left| \frac{\partial^k}{\partial \varepsilon^k} \left( \tilde{G}_{2,1}(x, \varepsilon) - \varepsilon^{m-1} \ln \varepsilon \hat{G}_{0,1}(x) \right) \right| \lesssim \sum_{\substack{i+j+l=k \\ i < m-1}} |\varepsilon|^{m-1-i-\delta} \langle x \rangle^{1+j} \|\langle x \rangle^{2+l} \psi_1(x)\|_1 \\ \lesssim \langle x \rangle^{1+k} \|\langle x \rangle^{2+k} \psi_1(x)\|_1$$

$$(101) \quad \left| \frac{\partial^m}{\partial \varepsilon^m} \left( \tilde{G}_{2,1}(x, \varepsilon) - \varepsilon^{m-1} \ln \varepsilon \hat{G}_{0,1}(x) \right) \right| \lesssim \sum_{\substack{i+j+l=m \\ i < m}} |\varepsilon|^{m-1-i-\delta} \langle x \rangle^{1+j} \|\langle x \rangle^{2+l} \psi_1(x)\|_1 \\ \lesssim |\varepsilon|^{-\delta} \langle x \rangle^{1+m} \|\langle x \rangle^{2+m} \psi_1(x)\|_1$$

and it can be similarly shown that for  $0 \leq k \leq m$

$$\left| \frac{\partial^k}{\partial \varepsilon^k} \left( \tilde{G}_{1,2}(x, \varepsilon) - \varepsilon^{m-1} \ln \varepsilon \hat{G}_{1,0}(x) \right) \right| \lesssim (1 + |\varepsilon|^{m-k-\delta}) \langle x \rangle^{2+k} \|\langle x \rangle^{1+k} \psi_1(x)\|_1$$

Also for  $0 \leq k \leq m$

$$(102) \quad \left| \frac{\partial^k}{\partial \varepsilon^k} \tilde{G}_{1,1}(x, \varepsilon) \right| \lesssim \sum_{i+j+l=k} \frac{|\varepsilon|^{2m-2-i} (|\ln \varepsilon| + 1)^2 \langle x \rangle^{2+j}}{1 + |\varepsilon| \langle x \rangle} \left\| \frac{\langle x \rangle^{2+l} \psi_1(x)}{1 + |\varepsilon| \langle x \rangle} \right\|_1 \\ \lesssim \langle x \rangle^{1+k} \|\langle x \rangle^{1+k} \psi_1(x)\|_1$$

$$(103) \quad \left| \frac{\partial^k}{\partial \varepsilon^k} \tilde{G}_{2,2}(x, \varepsilon) \right| \lesssim \sum_{i+j+l=k} (|\varepsilon|^{m-i-\delta} + 1) \langle x \rangle^{1+j} \|\langle x \rangle^{1+l} \psi_1(x)\|_1$$

Using (100)-(103) as well as (87) we have

$$(104) \quad \mathcal{G}(\psi_1) = \frac{r_+(\varepsilon) r_-(\varepsilon) \sum_{k,j=1,2} \tilde{G}_{j,k}(x, \varepsilon)}{W(\varepsilon)} = r_3(x) \varepsilon^{m-1} \ln \varepsilon + \hat{R}(x, \varepsilon)$$

where  $r_3(x) = (\hat{G}_{0,1}(x) + \hat{G}_{1,0}(x))/W(0)$  satisfies (95) and (97) holds with  $\hat{R}$  replacing  $R$ . (104) and (94) give

$$(105) \quad R(x, \varepsilon) - \hat{R}(x, \varepsilon) = r_3(x) \varepsilon^{m-1} \ln \varepsilon \left( 1 - \frac{1}{(1 + \varepsilon \langle x \rangle)^{m+2}} \right) - \mathcal{G}_0(\psi_1)$$

Now by direct calculation (97) holds with the right side of (105) replacing  $R$ , using

$$\left| \int_{\pm\infty}^x (u-x)^k e^{\mp \varepsilon(u-x)} \psi_1(u) du \right| \lesssim \langle x \rangle^k \|\langle x \rangle^k \psi_1(x)\|_1$$

for the estimates involving  $\mathcal{G}_0(\psi)$ . Thus (94) is valid for  $|\varepsilon| \leq 1$ .

For  $|\varepsilon| \geq 1$ , we analyze the numerator of (93) by noting that (recall that  $f_\pm = 1 + s_\pm$ )

$$(106) \quad f_-(u; \varepsilon) \int_{-\infty}^x e^{-\varepsilon(u-x)} f_+(u; \varepsilon) \psi_1(u) du = \int_{-\infty}^x e^{-\varepsilon(u-x)} \psi_1(u) du \\ + s_-(x; \varepsilon) \int_{-\infty}^x e^{-\varepsilon(u-x)} f_+(u; \varepsilon) \psi_1(u) du + \int_{-\infty}^x e^{-\varepsilon(u-x)} s_+(u; \varepsilon) \psi_1(u) du$$

and  $s_{\pm}$  satisfies (84). The same type of expansion works for the integral from  $-\infty$  to  $x$ . Thus by (93)

$$\mathcal{G}(\psi_1) = \frac{2\varepsilon\mathcal{G}_0(\psi_1) + (1+\varepsilon)R_3(x;\varepsilon)}{W(\varepsilon)}$$

for some  $R_3$  satisfying estimates of the type (97). Note that  $W(\varepsilon)$  satisfies (88). Obviously (97) holds with  $r_3(x)\frac{\varepsilon^{m-1}\ln\varepsilon}{(1+\varepsilon\langle x\rangle)^{m+2}}$  replacing  $R$ . Thus (94) is valid for  $|\varepsilon| \geq 1$  as well.  $\square$

Similarly we have

**Proposition 12.** *For  $\varepsilon \in \overline{\mathbb{H}} \setminus \{0\}$  we have*

$$\mathcal{G}(\varepsilon\psi_0) = r_4(x)\frac{\varepsilon^m \ln \varepsilon}{(1+\varepsilon\langle x\rangle)^{m+3}} - \frac{\psi_0(x)\varepsilon}{(\varepsilon+1)^2} + \frac{\varepsilon}{\varepsilon+1}\mathcal{G}_0(\psi_0(x)) + \frac{\varepsilon}{\varepsilon+1}\mathcal{G}_1(\psi'_0(x)) + R_4(x;\varepsilon)$$

where

$$(107) \quad \|\langle x \rangle^{-2}r_4(x)\|_{\infty} \lesssim \|\langle x \rangle^2\psi_0(x)\|_1$$

$$\|\psi_0(x)\|_{\infty} \lesssim \|\psi'_0(x)\|_1$$

$$\mathcal{G}_1(\psi) := \frac{1}{2(1+\varepsilon)} \left( \int_{\infty}^x e^{-\varepsilon(u-x)}\psi(u)du + \int_{-\infty}^x e^{\varepsilon(u-x)}\psi(u)du \right)$$

$$(108) \quad \|\langle x \rangle^{-k-2}\frac{\partial^k}{\partial \varepsilon^k}R_4(x;\varepsilon)\|_{\infty} \lesssim (1+|\varepsilon|)^{-2}(\|\langle x \rangle^{k+2}\psi_0(x)\|_1 + \|\langle x \rangle^{k+2}\psi'_0(x)\|_1) \quad (0 \leq k \leq m)$$

$$\|\langle x \rangle^{-m-3}\frac{\partial^{m+1}}{\partial \varepsilon^{m+1}}R_4(x;\varepsilon)\|_{\infty} \lesssim (|\varepsilon|^{\delta} + |\varepsilon|)^{-2}(\|\langle x \rangle^{m+3}\psi_0(x)\|_1 + \|\langle x \rangle^{m+3}\psi'_0(x)\|_1)$$

*Proof.* The proof is essentially the same as that of Proposition 11. In the case  $|\varepsilon| \leq 1$ , we simply use (98)-(103) with  $\psi_1$  replaced by  $\varepsilon\psi_0$  for  $0 \leq k \leq m+1$ , which gives the counterpart of (104) as

$$(109) \quad \mathcal{G}(\varepsilon\psi_0) = r_4(x)\varepsilon^m \ln \varepsilon + \hat{R}_4(x, \varepsilon)$$

where  $r_4$  satisfies (107) and  $\hat{R}_4$  satisfies the same estimates as  $R_4$  (see (108)). The rest follows in the same way as (105). Note that we have the obvious inequality  $|\psi_0(x)| \lesssim \|\psi'_0(x)\|_1$ .

For  $|\varepsilon| \geq 1$  we use integration by parts to get

$$(110) \quad \begin{aligned} \varepsilon e^{\varepsilon x} \int_{\infty}^x y_+(u)\psi_0(u)du &= \varepsilon e^{\varepsilon x} \int_{\infty}^x e^{-(\varepsilon+1)u}e^u(1+s_+(u;\varepsilon))\psi_0(u)du \\ &= -\frac{\varepsilon}{\varepsilon+1}(1+s_+(x;\varepsilon))\psi_0(x) + \frac{\varepsilon}{\varepsilon+1}e^{\varepsilon x} \int_{\infty}^x e^{-\varepsilon u}(\psi'_0(u) + \psi_0(u))du \\ &\quad + \frac{\varepsilon}{\varepsilon+1}e^{\varepsilon x} \int_{\infty}^x e^{-\varepsilon u}(s'_+(u;\varepsilon)\psi_0(u) + s_+(u;\varepsilon)(\psi'_0(u) + \psi_0(u)))du \end{aligned}$$

Similarly

$$(111) \quad \begin{aligned} -\varepsilon e^{-\varepsilon x} \int_{-\infty}^x y_-(u)\psi_0(u)du &= -\frac{\varepsilon}{\varepsilon+1}(1+s_-(x;\varepsilon))\psi_0(x) + \frac{\varepsilon}{\varepsilon+1}e^{-\varepsilon x} \int_{-\infty}^x e^{\varepsilon u}(\psi'_0(u) - \psi_0(u))du \\ &\quad + \frac{\varepsilon}{\varepsilon+1}e^{-\varepsilon x} \int_{-\infty}^x e^{\varepsilon u}(s'_-(u;\varepsilon)\psi_0(u) + s_-(u;\varepsilon)(\psi'_0(u) - \psi_0(u)))du \end{aligned}$$

Since  $s_{\pm}$  satisfy (84), by (93), (110) and (111) we have

$$\mathcal{G}(\varepsilon\psi_0) = -\frac{2\varepsilon\psi_0(x)}{(\varepsilon+1)W(\varepsilon)} + \frac{2\varepsilon\mathcal{G}_0(\psi_0(x)) + \mathcal{G}_1(\psi'_0(x))}{W(\varepsilon)} + \frac{\varepsilon\tilde{R}_3(x;\varepsilon)}{W(\varepsilon)}$$

for some  $\tilde{R}_3$  satisfying estimates of the type (108). The rest follows from (88).  $\square$



5. TIME DECAY OF  $\psi$ 

*Proof of Theorem 1.* We focus on the case  $\frac{\sin(t\sqrt{A})}{\sqrt{A}}\psi_1$  since the proof for  $\cos(t\sqrt{A})\psi_0$  is analogous.

We have using the decomposition (94) and that

$$(112) \quad \psi(x, t) = \mathcal{L}^{-1}\hat{\psi}(x, \varepsilon) \\ = \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} e^{\varepsilon t} \left( r_3(x) \frac{\varepsilon^{m-1} \ln \varepsilon}{(1 + \varepsilon \langle x \rangle)^{m+1}} + R(x; \varepsilon) \right) d\varepsilon + \mathcal{L}^{-1}\mathcal{G}_0(\psi_1(x))$$

where the inverse Laplace transform  $\mathcal{L}^{-1}$  can be represented using the Bromwich integral formula for the terms  $r_3(x) \frac{\varepsilon^{m-1} \ln \varepsilon}{(1 + \varepsilon \langle x \rangle)^{m+1}}$  and  $R$ , and we have

**Lemma 13.** *We have the estimate*

$$(113) \quad |\mathcal{L}^{-1}\mathcal{G}_0(\psi_1(x))| \lesssim (\langle x \rangle / \langle t \rangle)^{m+1} \|\langle x \rangle^{m+1} \psi_1(x)\|_1$$

*Proof.* By direct calculation

$$\mathcal{G}_0(\psi_1(x)) = \mathcal{L} \left( \frac{1}{2} \int_{t+x}^x e^{-t+u-x} \psi_0(u) du - \frac{1}{2} \int_{x-t}^x e^{-t-u+x} \psi_0(u) du \right)$$

The estimate (113) follows by integration by parts. For instance we have

$$(114) \quad \int_{t+x}^x e^{-t+u-x} \psi_0(u) du = \frac{e^{-t+u-x}}{(\sqrt{u^2+1})^{m+1}} \Big|_{t+x}^x \int_{t+x}^x (\sqrt{u^2+1})^{m+1} \psi_0(u) du \\ - \int_{t+x}^x \left( \frac{e^{-t+u-x}}{(\sqrt{u^2+1})^{m+1}} \right)' \left( \int_0^u (\sqrt{v^2+1})^{m+1} \psi_0(v) dv \right) du$$

where the first term on the right side can be estimated using the elementary inequality

$$\frac{1}{(\sqrt{(t+x)^2+1})^{m+1}} \lesssim (\langle x \rangle / \langle t \rangle)^{m+1}$$

and the last term can be estimated by noting that

$$(115) \quad \left| \int_{t+x}^x \left( \frac{e^{-t+u-x}}{(\sqrt{u^2+1})^{m+1}} \right)' du \right| \leq \left| \int_{t+x}^x \frac{e^{-t+u-x}}{(\sqrt{u^2+1})^{m+1}} du \right| \leq \int_x^{x+t/2} \frac{e^{-t+u-x}}{(\sqrt{u^2+1})^{m+1}} du \\ + \int_{x+t/2}^{x+t} \frac{e^{-t+u-x}}{(\sqrt{u^2+1})^{m+1}} du \lesssim e^{-t/2} + \sup_{t/2 \leq v \leq t} \frac{1}{(\sqrt{(x+v)^2+1})^{m+1}} \lesssim (\langle x \rangle / \langle t \rangle)^{m+1}$$

□

To estimate the other terms of (112) note that by contour deformation and Watson's Lemma we have for large  $t$

$$(116) \quad \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} e^{\varepsilon t} r_3(x) \frac{\varepsilon^{m-1} \ln \varepsilon}{(1 + \varepsilon \langle x \rangle)^{m+1}} d\varepsilon = - \int_0^{-1/2} e^{\varepsilon t} \frac{r_3(x) \varepsilon^{m-1}}{(1 + \varepsilon \langle x \rangle)^{m+1}} d\varepsilon + O(e^{-t/2}) r_3(x) \\ = (-1)^{m+1} (m-1)! r_3(x) t^{-m} (1 + O(t^{-1}) \langle x \rangle) + O(e^{-t/2}) r_3(x)$$

Note that the left side of (116) is obviously bounded by  $\text{const.} |r_3(x)|$  uniformly for all  $t \geq 0$  and  $r_3$  satisfies (95). Finally by integration by parts and (97) we have for  $t > 0$

$$\left| \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} e^{\varepsilon t} R(x; \varepsilon) d\varepsilon \right| \lesssim t^{-m} \left| \int_{-\infty i}^{\infty i} e^{\varepsilon t} \frac{\partial^m R(x; \varepsilon)}{\partial \varepsilon^m} d\varepsilon \right|$$

Since  $R(x; \varepsilon)$  and  $\frac{\partial^m R(x; \varepsilon)}{\partial \varepsilon^m}$  satisfies (97), we have for  $t \geq 0$

$$(117) \quad \|\langle x \rangle^{-m-2} \mathcal{L}^{-1} R(x; \varepsilon)\|_{\infty} \lesssim \langle t \rangle^{-m} \|\langle x \rangle^{m+2} \psi_1(x)\|_1$$

Furthermore, it follows from the Riemann-Lebesgue lemma that

$$(118) \quad \lim_{t \rightarrow \infty} t^m \mathcal{L}^{-1} R(x; \varepsilon) = 0$$

We define  $\hat{r}_1(x) = (-1)^{m+1}(m-1)!r_3(x)$  and

$$R_1(x, t) = \langle t \rangle^m \mathcal{L}^{-1} \left( \mathcal{G}_0(\psi_1(x)) + R(x; \varepsilon) + r_3(x) \frac{\varepsilon^{m-1} \ln \varepsilon}{(1 + \varepsilon \langle x \rangle)^{m+1}} - \hat{r}_1(x) \right)$$

The result for  $\frac{\sin(t\sqrt{A})}{\sqrt{A}}\psi_1$  in Theorem 1 then follows from (112) using (113), (116), (117), and (118); for  $\cos(t\sqrt{A})\psi_0$  the estimates follow from Proposition 12 in a similar way.  $\square$

### 5.1. Genericity of decay rate.

**Remark 6.** *In view of the definition of  $r_3$  (below (104)) it is clear that  $r_3$  is nonzero for generic initial condition  $\psi_1$ , meaning the time decay  $t^{-m}$  is generic.*

## 6. MORE GENERAL POTENTIALS

**6.1. Sums of inverse powers.** Assume  $V(x) = \text{const.} x^{-\alpha_1^\pm} (1 + \sum_{k=1}^n a_k^\pm x^{-\beta_k^\pm})$  for large  $\pm x$  where  $\alpha_1^\pm > 2$  and  $\beta_k^\pm > 0$ .

Without loss of generality we study large  $x$ . Now (8) has the form

$$(119) \quad F(\tau) = \text{const.} (\varepsilon \tau)^{\alpha_1^+ - 1} (1 + \sum_{k=1}^n b_k(\varepsilon \tau)^{\beta_k^+}) \\ + \text{const.} \varepsilon^{\alpha_1^+ - 2} \int_0^\tau (\tau - u)^{\alpha_1^+ - 1} (1 + \sum_{k=1}^n b_k(\varepsilon(\tau - u))^{\beta_k^+}) \frac{F(u)}{u(u+2)} du$$

It can be shown that for large  $\tau$

$$(120) \quad \frac{F(\tau)}{\varepsilon^2 \tau(\tau+2)} = \text{const.} (\varepsilon \tau)^{\alpha_1^+ - 3} (1 + \sum_{k=1}^n b_k(\varepsilon \tau)^{\beta_k^+}) \left( 1 + \sum_{k=0}^n \sum_{l=1}^\infty \sum_{m=0}^l c_{klm} (\varepsilon \tau)^{l(\alpha_1^+ - 2 + \beta_k^+)} (\ln \tau)^m \right)$$

where  $\beta_0 = 0$  and  $\ln$  terms are only present for  $\alpha_1^+ \in \mathbb{N}$ .

The counterpart of (34) is now the expansion

$$s(x; \varepsilon) = \varepsilon^{\alpha_1^+ - 2} \sum_{k=0}^n (\tilde{c}_k(x) + \varepsilon c_k(x)) \varepsilon^{\beta_k^+} + \varepsilon^{\alpha_1^+ - 2} \ln \varepsilon \sum_{k=0}^n (\tilde{C}_k(x) + \varepsilon C_k(x)) \varepsilon^{\beta_k^+} + \dots$$

where terms with higher orders of  $\ln$  are omitted and the  $\ln$  terms are only present for  $\alpha_1^+ \in \mathbb{N}$ .

This implies the counterpart of the expansion (78)

$$y_+(x; \varepsilon) = r(\varepsilon) e^{-\varepsilon x} (\hat{D}_1(x) \varepsilon^{\alpha_1^+ - 1} + \hat{D}_2(x) \varepsilon^{\alpha_1^+ - 1} \ln \varepsilon + R(x; \varepsilon))$$

where  $\frac{\partial^k R(x; \varepsilon)}{\partial \varepsilon^k}$  are bounded by  $\varepsilon^{\delta-1}$  for  $0 \leq k \leq \lceil \alpha_1^+ \rceil$  and some  $\delta > 0$ , and  $D_2 = 0$  except for  $\alpha_1 \in \mathbb{N}$ .

Arguments similar to those showing Proposition 11 lead to the same type of expansion for  $\hat{\psi}$ , which then implies that

$$\frac{\sin(t\sqrt{A})}{\sqrt{A}} \psi_1 \sim \hat{r}_1(x) t^{-\alpha} \\ \cos(t\sqrt{A}) \psi_0 \sim \hat{r}_0(x) t^{-\alpha-1}$$

The detailed estimates for the higher order remainder as in Theorem 1 can be obtained in similar ways.

**6.2. Inverse power with higher order correction.** Here we discuss the general  $m$  analog of potentials of the type in [6]. Assume  $V(x) = V_0(x) + V_1(x)$  and  $V_0(x) = c_v/x^m$  for  $|x| \geq x_+$ ,  $V_1$  is piecewise continuous,  $|V_1^{(k)}(x)| \lesssim \langle x \rangle^{-m-k-\delta_1}$  where  $0 \leq k \leq m+2$  and  $\delta_1 > 3$ . One can show that for  $x \geq x_+$  and  $|\varepsilon| \leq 1/x$ ,  $\tilde{y}_1$  has the expansion

$$(121) \quad \tilde{y}_1(x; \varepsilon) = r(\varepsilon) \left( \tilde{B}_0(x) + \tilde{B}_1(x) \varepsilon^{m-1} \ln \varepsilon + f_d(x, \varepsilon) \right)$$

where  $\tilde{B}_k$  solves  $f''(x) = V(x)f(x)$  with  $\tilde{B}_0 \sim \Phi_1 =: B_0$ ,  $\tilde{B}_1 \sim B_1$  for large  $x$  and  $f_d(x, \varepsilon)$  satisfies (57).

Indeed, by (55) and standard ODE analysis one can find  $\tilde{B}_k(x)$  with  $\tilde{B}_k''(x) = V(x)\tilde{B}_k(x)$ ,  $|\tilde{B}_k(x) - B_k(x)| \lesssim \langle x \rangle^{2-m-\delta_1} \langle B_k(x) \rangle$  and  $|\tilde{B}_k'(x) - B_k'(x)| \lesssim \langle x \rangle^{1-m-\delta_1} \langle B_k'(x) \rangle$ .

By (55) we have

$$(122) \quad f_+(x; \varepsilon) = r(\varepsilon) \left( \hat{\Phi}_1(x) + B_1(x) \varepsilon^{m-1} \ln \varepsilon + f_c(x, \varepsilon) \right) =: r(\varepsilon) \left( \phi(x; \varepsilon; B) + f_c(x, \varepsilon) \right)$$

where  $f_c$  satisfies (57) and

$$f_c''(x, \varepsilon) - 2\varepsilon f_c'(x, \varepsilon) - V_0(x)f_c(x, \varepsilon) = 2\varepsilon \phi(x; \varepsilon; B')$$

Similarly we write  $\tilde{f}_1(x; \varepsilon) = r(\varepsilon) \left( \phi(x; \varepsilon; \tilde{B}) + f_c(x, \varepsilon) + g_1(x, \varepsilon) \right)$ , which implies  $\tilde{g}_1(x, \varepsilon) = e^{-\varepsilon x} g_1(x, \varepsilon)$  satisfies the equation

$$\tilde{g}_1''(x, \varepsilon) - \varepsilon^2 \tilde{g}_1(x, \varepsilon) - V_0(x) \tilde{g}_1(x, \varepsilon) = e^{-\varepsilon x} \phi_1(x, \varepsilon) + V_1(x) \tilde{g}_1(x, \varepsilon)$$

where  $\phi_1(x, \varepsilon) = 2\varepsilon \phi(x; \varepsilon; \tilde{B}' - B') + \varepsilon V_1(x) \hat{f}_c(x, \varepsilon)$  and  $\hat{f}_c = f_c/\varepsilon$ . Equivalently we have the integral equation

$$(123) \quad g_1(x) = \mathcal{G}(e^{-\varepsilon x} \phi_1(x; \varepsilon)) + \mathcal{G}(V_1(x) g_1(x))$$

Using Proposition 10 we see that  $|\phi_1(x; \varepsilon)| \lesssim |\varepsilon| \langle x \rangle^{1-m-\delta_1}$  for all  $x \geq x_+$  and thus (123) is contractive under the norm  $\|\langle x \rangle^{m-2+\delta_1} g_1(x)\|_\infty$ , and by taking derivatives of (123) we have

$$\left| \frac{\partial^k g_1(x; \varepsilon)}{\partial \varepsilon^k} \right| \lesssim x^{2-m-\delta_1+k} \quad (0 \leq k \leq m-1); \quad \left| \frac{\partial^k g_1(x; \varepsilon)}{\partial \varepsilon^k} \right| \lesssim |\varepsilon|^{m-k-\delta} x^{2-\delta_1} \quad (m \leq k \leq m+1)$$

since

$$\left| e^{\varepsilon x} \int_\infty^x e^{-2\varepsilon t} t^k \phi_1(t; \varepsilon) dt \right| \lesssim 1$$

if  $k - m - \delta_1 < -2$ . Thus if  $\delta_1 > 3$  then the counterpart of Proposition 7 holds. The rest of the proof is similar.

## 7. APPENDIX

For completeness, in this section we provide a short self-contained proof justifying the use of the Laplace transform.

**Proposition 14.** *Assume the initial conditions  $f(x) = u(x, 0)$  and  $g(x) = u_t(x, 0)$  are in  $L^1(\mathbb{R})$  and  $V \in L^\infty(\mathbb{R})$ . Then, if  $\nu > \sqrt{2}\|V\|_\infty^{\frac{1}{2}}$  we have  $\sup_{t>0} e^{-\nu t} \|u(t, \cdot)\|_1 < \infty$ , and thus  $u(t, x)$  is Laplace transform in  $t$ .*

*Proof.* We use the Duhamel principle to write (1) in the form

$$(124) \quad u = \mathcal{A}u; \quad \mathcal{A}u := \frac{f(x-t) + f(x+t)}{2} + \frac{1}{2} \int_{-\infty}^{\infty} \chi_t(y-x) g(y) dy + \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} u(y, s) V(y) \chi_{t-s}(y-x) dy ds$$

where  $\chi_a$  is the characteristic function of the interval  $[-a, a]$ . Consider the Banach space

$$(125) \quad \mathcal{B} = \{u \in C(\mathbb{R}) \mid \|u\|_\nu := \sup_{t \in \mathbb{R}^+} e^{-\nu t} \|u(t, \cdot)\|_1 < \infty\}; \quad (\nu > \sqrt{2}\|V\|_\infty^{\frac{1}{2}})$$

Applying Fubini to integrate first in  $x$ , we see that  $\|\int_{-\infty}^{\infty} \chi_t(y-x)g(y)dy\|_1 \leq 2t\|g\|_1$  and (since by definition  $\|u(\cdot, s)\|_1 \leq \|u\|_{\nu}e^{\nu s}$ )

$$(126) \quad \sup_{t>0} e^{-\nu t} \left\| \int_0^t \int_{-\infty}^{\infty} u(y, s)V(y)\chi_{t-s}(y-x)dyds \right\|_1 \\ \leq \|V\|_{\infty}\|u\|_{\nu} \sup_{t>0} e^{-\nu t} \int_0^t 2(t-s)e^{\nu s}ds \leq 2\|V\|_{\infty}\nu^{-2}\|u\|_{\nu}$$

Using (126) we see that  $\mathcal{A} : \mathcal{B} \rightarrow \mathcal{B}$  is contractive. Also, assuming  $f, g$  and  $V$  are smooth, the solution is seen to be smooth too: since  $u \in L^1$ , Duhamel's formula shows that it is continuous; then, as usual, using continuity we derive differentiability, and inductively, we see that  $u$  is smooth.  $\square$

*Proof of Lemma 6.* This is by straightforward calculation. For  $n \geq 0$  we get

$$(127) \quad \int_3^{\infty} e^{-\varepsilon\tau x} \tau^n (\ln \tau)^l d\tau = \left( \int_0^{\infty} - \int_0^3 \right) e^{-\varepsilon\tau x} \tau^n (\ln \tau)^l d\tau = \frac{1}{(\varepsilon x)^{n+1}} \int_0^{\infty} e^{-u} u^n (\ln u \\ - \ln(\varepsilon x))^l du - \int_0^3 e^{-\varepsilon\tau x} \tau^n (\ln \tau)^l d\tau = \frac{1}{(\varepsilon x)^{n+1}} \sum_{q=0}^l c_q^{(n;l)} (\ln(\varepsilon x))^q - \int_0^3 e^{-\varepsilon\tau x} \tau^n (\ln \tau)^l d\tau$$

where the last term is  $R_{a,n}$  and (45) is immediate, and

$$c_q^{(n;l)} = \left( \int_0^{\infty} e^{-u} u^n (\ln u)^{l-q} du \right) \frac{(-1)^{q!} l!}{(l-q)! q!}$$

In particular  $c_1^{(0;1)} = -1$ .

Now (46) for  $n = -1$  follows from integration by parts

$$(128) \quad \int_3^{\infty} e^{-\varepsilon\tau x} \tau^{-1} (\ln \tau)^l d\tau = -\frac{1}{l+1} e^{-3\varepsilon x} (\ln 3)^{l+1} + \frac{\varepsilon x}{l+1} \int_3^{\infty} e^{-\varepsilon\tau x} (\ln \tau)^{l+1} d\tau$$

where the first term satisfies (45), and the last integral in (128) was evaluated in (127).

For  $n < -1$  we have by integration by parts

$$(129) \quad \int_3^{\infty} e^{-\varepsilon\tau x} \tau^n (\ln \tau)^l d\tau = -\frac{1}{n+1} e^{-3\varepsilon x} 3^{n+1} (\ln 3)^l \\ - \frac{l}{n+1} \int_3^{\infty} e^{-\varepsilon\tau x} \tau^n (\ln \tau)^{l-1} d\tau + \frac{\varepsilon x}{n+1} \int_3^{\infty} e^{-\varepsilon\tau x} \tau^{n+1} (\ln \tau)^l d\tau$$

and (46) follows by induction on  $l$  and  $n$  using (129) and integration by parts.  $\square$

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